Course Title:Early Bird Math Olympiad U   Paper: Combinatorics

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Q.1) A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called symmetric if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

(A) 510   **(B)** 1022   **(C)** 8190   **(D)** 8192   **(E)** 65,534

Q.2) In the expression (____ × ____) + (____ × ____) each blank is to be filled in with one of the digits $1, 2, 3$, or 4, with each digit being used once. How many different values can be obtained?

(A) 2   **(B)** 3   **(C)** 4   **(D)** 6   **(E)** 24

Q.3) How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

(A) 128   **(B)** 192   **(C)** 224   **(D)** 240   **(E)** 256

Q.4) A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

(A) 202   **(B)** 223   **(C)** 224   **(D)** 225   **(E)** 234

Q.5) At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?

(A) 240   **(B)** 245   **(C)** 290   **(D)** 480   **(E)** 490

Q.6) Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

(A) 12   **(B)** 16   **(C)** 28   **(D)** 32   **(E)** 40

Q.7) How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property.

(A) 226   (B) 243   (C) 270   (D) 469   (E) 486

Q.8) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

(A) 1   (B) 2   (C) 3   (D) 4   (E) 5

Q.9) How many of the base-ten numerals for the positive integers less than or equal to 2017 contain the digit 0?

(A) 469   (B) 471   (C) 475   (D) 478   (E) 481

Q.10) In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?
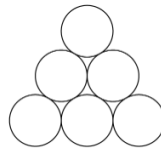
(A) 6   (B) 8   (C) 9   (D) 12   (E) 15

Q.11) Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

(A) 1024   (B) 1524   (C) 1533   (D) 1536   (E) 2048

Q.12) Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

(A) 1   (B) 3   (C) 6   (D) 12   (E) 24

Q.13) Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions "left" and "right" are from the point of view of the people as

they sit in the seats.) During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

(A) 1   **(B)** 2   **(C)** 3   **(D)** 4   **(E)** 5

Q.14) Erin the ant starts at a given corner of a cube and crawls along exactly 7 edges in such a way that she visits every corner exactly once and then finds that she is unable to return along an edge to her starting point. How many paths are there meeting these conditions?

(A) 6   **(B)** 9   **(C)** 12   **(D)** 18   **(E)** 24

Q.15) Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?

(A) 2   **(B)** 3   **(C)** 4   **(D)** 5   **(E)** 6

Q.16) The numbers $1, 2, 3, 4, 5$ are to be arranged in a circle. An arrangement is *bad* if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

(A) 1   **(B)** 2   **(C)** 3   **(D)** 4   **(E)** 5

Q.17) A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

(A) 6   **(B)** 8   **(C)** 9   **(D)** 12   **(E)** 16

Q.18) A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

(A) 10   **(B)** 12   **(C)** 15   **(D)** 18   **(E)** 25

Q.19) Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?

(A) 540   **(B)** 600   **(C)** 720   **(D)** 810   **(E)** 900

Q.20) All 20 diagonals are drawn in a regular octagon. At how many distinct points in the interior of the octagon (not on the boundary) do two or more diagonals intersect?

(A) 49   **(B)** 65   **(C)** 70   **(D)** 96   **(E)** 128

Q.21) Chubby makes nonstandard checkerboards that have 31 squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

(A) 480   **(B)** 481   **(C)** 482   **(D)** 483   **(E)** 484

Q.22) Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

(A) 60   (B) 170   (C) 290   (D) 320   (E) 660

Q.23) A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

(A) 729   **(B)** 972   **(C)** 1024   **(D)** 2187   **(E)** 2304

Q.24) In a round-robin tournament with 6 teams, each team plays one game against each other team, and each game results in one team winning and one team losing. At the end of the tournament, the teams are ranked by the number of games won. What is the maximum number of teams that could be tied for the most wins at the end on the tournament?

(A) 2   **(B)** 3   **(C)** 4   **(D)** 5   **(E)** 6

Q.25) Let $(a_1, a_2, \dots a_{10})$ be a list of the first 10 positive integers such that for each $2 \leq i \leq 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

(A) 120   **(B)** 512   **(C)** 1024   **(D)** 181,440   **(E)** 362,880

Q.26) Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

(A) 108   **(B)** 132   **(C)** 671   **(D)** 846   **(E)** 1105

Q.27) How many even integers are there between 200 and 700 whose digits are all different and come from the set 1,2,5,7,8,9?

(A) 12   (B) 20   (C) 72   (D) 120   (E) 200

Q.28) Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

(A) 2520   **(B)** 2880   **(C)** 3120   **(D)** 3250   **(E)** 3750

Q.29) Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

(A) 1930   **(B)** 1931   **(C)** 1932   **(D)** 1933   **(E)** 1934

Q.30) Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20   (B) 40   (C) 60   (D) 160   (E) 320

Q.31) Ten chairs are evenly spaced around a round table and numbered clockwise from 1 through 10. Five married couples are to sit in the chairs with men and women alternating, and no one is to sit either next to or across from his/her spouse. How many seating arrangements are possible?

(A) 240   (B) 360   (C) 480   (D) 540   (E) 720

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Q.32) How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

(A)128   (B)192   (C)224   (D)240   (E)256

Q.33) A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values n, $9 \le n \le 2017$, can be the number of participants?

(A) 477   **(B)** 482   **(C)** 487   **(D)** 557   **(E)** 562

Q.34) A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

(A) 385   **(B)** 665   **(C)** 945   **(D)** 1140   **(E)** 1330

Q.35) Six chairs are evenly spaced around a circular table. One person is seated in each chair. Each person gets up and sits down in a chair that is not the same chair and is not adjacent to the chair he or she originally occupied, so that again one person is seated in each chair. In how many ways can this be done?

(A) 14   **(B)** 16   **(C)** 18   **(D)** 20   **(E)** 24

Q.36) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100   **(B)** 2220   **(C)** 3000   **(D)** 3120   **(E)** 3125

Q.37) Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

(A) 96   **(B)** 108   **(C)** 156   **(D)** 204   **(E)** 372

Q.38) Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \leq i \leq 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

(A) 120   **(B)** 512   **(C)** 1024   **(D)** 181,440   **(E)** 362,880

Q.39) How many positive integers less than 1000 are 6 times the sum of their digits?

(A) 0   **(B)** 1   **(C)** 2   **(D)** 4   **(E)** 12

Q.40) Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

(A) 89   **(B)** 90   **(C)** 120   **(D)** 2^{10} **(E)** $2^2 3^8$