STUDY MATERIAL FOR AMC 8 AND MATHCOUNTS

Reading Note

Cheenta Creators Team

Day - 6

Topic - Geometry
Transformations and Symmetries

0.1 Lesson Overview

We shall begin this lesson by introducing the concept of a transformation of a plane. We shall then define symmetries in terms of these transformations.

0.2 Transformations of the plane

Any function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the plane. Certain transformations carry a geometric meaning, and they can help us solve problems in elementary Geometry.

0.3 Isometries

Isometries are a special class of transformations of the plane that preserve the distance between points. In other words, if $d(p, q)$ denotes the distance between the two arbitrary points p, q, then we always have

$$d(f(p), f(q)) = d(p, q)$$

for an isometry f.

> **Think!** Can you give an example of a transformation of the plane which is not an isometry?

There exist three basic isometries, as explained below.

- **Translation**

  A translation is a function of the form $T(x, y) = (x + a, y + b)$ for fixed real numbers a, b. Geometrically speaking, the transformation 'pushes' all points by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

- **Rotation**

  A rotation is defined by two parameters, a point $P(a, b)$ (the centre) and an angle ϕ. Geometrically speaking, under such a transformation the whole plane is rotated counterclockwise by the angle ϕ except for the point P. Note that, a rotation can be composed with a translation to get another rotation. The only thing that changes is the centre.

  To give an expression for a rotation map, it is helpful to work in terms of polar coordinates instead of cartesian coordinates. Given any point Q in the plane with origin O, we identify it with a pair of real numbers (r, θ), where $r = |OQ|$ and θ is the angle that $\overrightarrow{OQ}$ makes with the positive X axis. Under this identification, a rotation about O by an angle ϕ is simply the map $(r, \theta) \mapsto (r, \theta + \phi)$.

  > **Think!** Can you find an expression for rotations about points different from the origin?

> **Challenge** : For which angles ϕ does there exist a natural number n such that R_ϕ^n = identity?

- **Reflection**

  A reflection is taken with respect to a fixed line ℓ in the plane. Given the line ℓ, the image of a point P is defined to be the unique point $R(P)$ such that ℓ is the perpendicular bisector of the segment $\overline{Pf(P)}$. Note that, $R(R(P)) = P$ so

  $$R^2 = \text{Identity}.$$

  When ℓ is the X axis, the reflection is given by

  $$(x, y) \mapsto (x, -y).$$

> **Think!** Can you find the expression for the reflection function for an arbitrary line $ax + by = c$?

0.3.1 Basics

We shall consider on the letters of the roman alphabet. Write "M" on paper and try to divide the letter into two parts by folding the paper. In the figure the blue line indicates the fold.

Now let us see some other letters, such as “H”, “W”, “O” and “A”.

Note that we may divide each of the letters into two identical parts by folding.
This is the idea of line of symmetry. If for a given figure we can imagine a line that divides the figure into two identical parts then that line is called **a line of symmetry** .
Obviously from the previous figures we can conclude that there can be 3 **types** of lines of symmetry. They are

1. **Vertical** (in case of the letter "M")

2. **Horizontal** (in case of the letter "H")

3. **Both** (in case of the letter "O")

Now the question is : what exactly is **symmetry**?

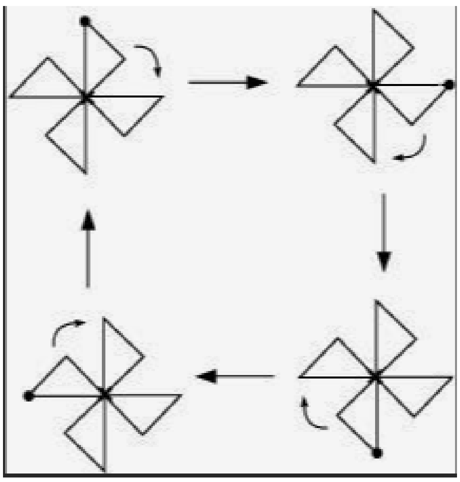
In geometry, an object has a symmetry if there is an isometry (such as translation, rotation or reflection) that maps the figure/object onto itself (i.e., the object has an invariance under the transform).

Think! Which of the following figures has the greatest number of lines of symmetry? (A) equilateral triangle (B) non-square rhombus (C) non-square rectangle (D) isosceles trapezoid (E) square

0.4 Types of symmetry

Rotation Symmetry

Look at this figure. Most children have played which such toys. This is one of the best examples of rotation symmetry.



Think! Exactly which rotations leave this figure invariant?

Reflection symmetry

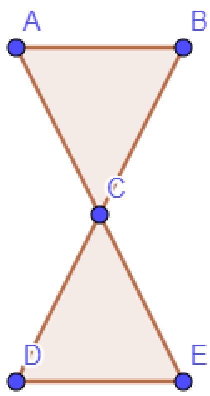
Reflection symmetry is symmetry with respect to a reflection. That is, a figure which does not change upon undergoing a reflection is said to have reflection symmetry. It is also called as mirror image symmetry.
Now think about the following image and try to explain why it has reflection symmetry.

Translation symmetry

Translation means displacement, and figures which remains unchanged after translations are said to have translation symmetry.
Figures having translation symmetry are necessarily infinite. For example, an infinite fence has translation symmetry along its length.

Point symmetry

Now see the figure. It is symmetric about the point "C".



> **Challenge** : Try to draw some figures which are symmetric about a point.

Last Challenge : For each category of symmetry try to find a couple of more examples.