STUDY MATERIAL FOR AMC 8 AND MATHCOUNTS

Reading Note

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Day - 3

Topic - Combinatorics
The Principle of Inclusion and Exclusion

0.1 Lesson Overview

The principle of inclusion and exclusion is the last of the basic counting principles. Due to the intricacies of the statement, a separate lesson has been dedicated to it.

0.2 The Principle of Inclusion and Exclusion

Sometimes in counting problems, we either undercount or overcount. Instead of trying to find direct ways to solve the problems, it is often easier to begin with a wrong count and account later for the error. Before we discuss an example, we must state the principle of inclusion and exclusion.

Theorem 0.1. *Given a collection $B_1, B_2, \cdots, B_n$ of finite sets, we have*

$$\left|\bigcup_{i=1}^{n} B_i\right| = \sum_{i=1}^{n} |B_i| - \sum_{i,j:1\le i<j\le n} \left|B_i \cap B_j\right| + \sum_{i,j,k:1\le i<j<k\le n} \left|B_i \cap B_j \cap B_k\right| - \cdots + (-1)^{n-1} |B_1 \cap \cdots \cap B_n|.$$

As the statement is written in very compact notation, it could be a good idea to try to break it down by looking at the small cases.

To begin with, let us assume that $n = 2$. Then the statement is equivalent to

$$|B_1 \cup B_2| = |B_1| + |B_2| - |B_1 \cap B_2|.$$

This should be trivial from a Venn diagram. Intuitively speaking, we want to count the number of elements in $B_1 \cup B_2$. The term $|B_1| + |B_2|$ represents an overcounting, because some terms occur both in B_1 and B_2. As these terms are counted twice, they must be subtracted once. This is reflected in the correction term $|B_1 \cap B_2|$.

Taking $B_2 = B \cup C$ for some sets B and C, we get

$$\begin{aligned} |B_1 \cup B \cup C| &= |B_1| + |B \cup C| - |B_1 \cap (B \cup C)| \\ &= |B_1| + |B| + |C| - |B \cap C| - |(B_1 \cap B) \cup (B_1 \cap C)| \\ &= |B_1| + |B| + |C| - |B \cap C| - |B_1 \cap B| - |B_1 \cap C| + |B_1 \cap B \cap C|. \end{aligned}$$

This formula is an example of correction being done is successive steps. The term $|B_1| + |B| + |C|$ is clearly an overcounting, so we need to account for elements that are repeated. In $|B_1| + |B| + |C| - |B \cap C| - |B_1 \cap B| - |B_1 \cap C|$, elements that are common between two of the sets are accounted for. However, here there is some undercounting. Elements that are common between all the sets have been added thrice and subtracted thrice. To add them back in, we must add the term $|B_1 \cap B \cap C|$. Now the count is complete.

Example 0.2. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Source : AMC 10B 2017, Problem 13.

Solution.
Let us denote

by A_1 the set of students taking yoga,

by A_2 the set of students taking bridge,

and by A_3 the set of students taking painting.

The information provided in the problem can be rewritten in set-theoretic language as

$$|A_1| = 10, |A_2| = 13, |A_3| = 9,$$

$$|A_1 \cup A_2 \cup A_3| = 20$$

and

$$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - 2|A_1 \cap A_2 \cap A_3| = 9.$$

The last equation requires some explanation. Indeed, in $|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|$, the students taking all the courses are counted thrice. To make up for this overcounting, we must subtract their number twice. This is the meaning of the term $-2|A_1 \cap A_2 \cap A_3|$.

We need to find $|A_1 \cap A_2 \cap A_3|$. The principle of inclusion and exclusion gives

$$\begin{aligned}
20 &= |A_1 \cup A_2 \cup A_3| \\
&= |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|) + |A_1 \cap A_2 \cap A_3| \\
&= (10 + 13 + 9) - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - 2|A_1 \cap A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3| \\
&= 23 - |A_1 \cap A_2 \cap A_3|.
\end{aligned}$$

Hence $|A_1 \cap A_2 \cap A_3| = 3$.