STUDY MATERIAL FOR AMC 8 AND MATHCOUNTS

Reading Note

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Day - 3

Topic - Geometry
Volume and Surface Area

0.1 Lesson Overview

In this lesson, we shall discuss the concept of volume and calculate the volumes of certain solids. Later, we shall also calculate the surface areas of those solids.

0.2 The volume

What is the volume of a three-dimensional figure? In fact, what is the area of a two dimensional figure, for that matter? These concepts are usually explained in terms of words, and rigourous mathematical language is not used. However, mathematicians are seldom satisfied with such explanations and they have their own way of looking at volumes (or areas).

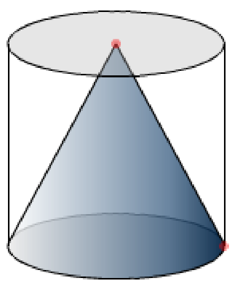
The rigourous way is the following : choose the simplest shape possible, which is a cuboid (or a rectangle, while in 2D). Define its volume as (length) × (width) × (height) (in case of 2D, it is (length) × (width)). Afterwards, try to approximate all other figures in terms of these basic building blocks.

Why is this better? It is because such an approach lets us keep the amount of intuition at a minimum. The only thing we 'assume' is the formula for the volume/area of the most basic shape. The rest follows from there.

Think! Can you find the area of a triangle/ a parallelogram/ a trapezium in this way?

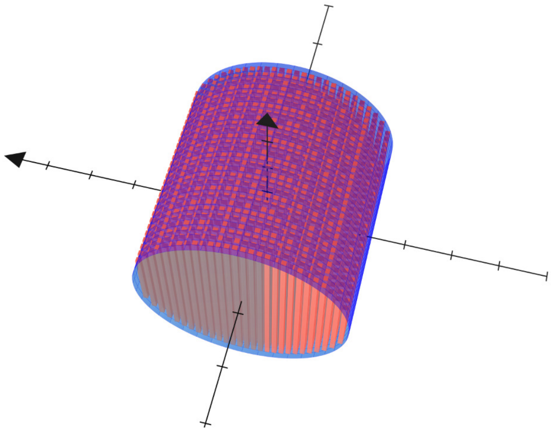
0.2.1 Cone and cylinder

It was Archimedes who first found the volumes of a cylinder and a cone in terms of their dimensions. The volume of a right circular cylinder with base radius r and height h is equal to $\pi r^2 h$. This can be intuitively guessed by imagining the cylinder to be a collection of infinitely many circles (say 'h' many !) stacked upon the base circle.



The formula for the volume of a right circular cone has a more interesting history. Archimedes put a cone inside a cylinder with the same base radius and height, and he compared their volumes. This was achieved by filling the cylinder to the brim with water and then measuring the volume of the water displaced by the cone. In this way, the volume of the cone was found to be equal to one-third of the volume of the cylinder. This means that the volume of a right circular cone with base radius r and height h is equal to $\frac{1}{3}\pi r^2 h$.

The 'correct' method, of course, would be to approximate both these objects with a collection of boxes. This approach lies at the heart of integral calculus, which is unfortunately beyond our scope.



0.2.2 Sphere

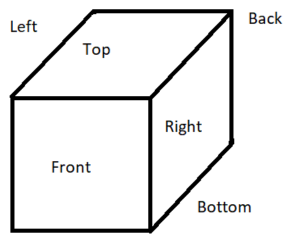
The volume of a sphere with radius r is equal to $\frac{4}{3}\pi r^3$.

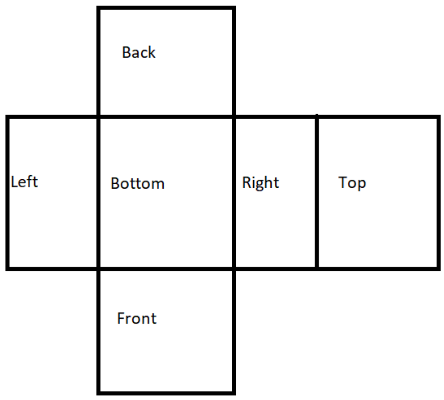
> **Long-term challenge** : Can you find a proof of these formulae? If not, can you at least find an explanation?

0.3 Surface area

0.3.1 Cuboid

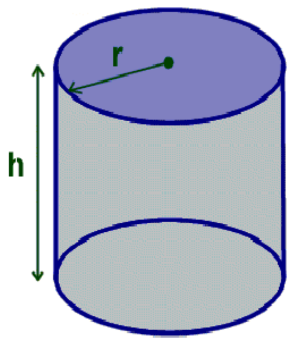
Given a cuboid with dimensions l, b and w, we may cut it open to get a collection of rectangles (as shown in the figure). The rectangles have dimensions $l \times b, b \times w$ and $w \times l$, and there are exactly two of each type. Hence the total area covered by them is $2(lb + bw + wl)$.





0.3.2 Cylinder

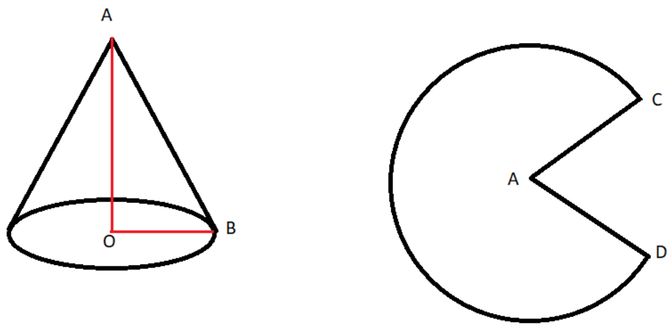
Suppose we have a right circular cylinder with base radius r and height h. Then the surface area is the sum of the areas of the discs with the area of the curved surface. The total area of the two discs is $2\pi r^2$. To get the area of the curved surface, we shall need to cut it open along a cut made perpendicular to the plane of the discs.



After cutting open, the curved surface becomes a rectangle with dimensions h and $2\pi r$. Hence its area is $2\pi rh$. Finally, the total surface area of the cylinder is given by $2\pi r(r+h)$.

0.3.3 Cone

A right circular cone has a curved surface and a planar surface. The planar surface being a disc, it is easy to calculate its area. However, calculating the area of the curved surface is a bit more tricky.



Let A be the apex of the cone and O be the centre of the base. Also, let B be an arbitrary point on the boundary of the base. Let $|AO| = h$ and $|OB| = r$. Then the pythagorean theorem gives that $AB = \sqrt{r^2 + h^2}$. Now we cut open the curved surface along the segment AB. The result is a sector of a circle with centre A. Also, the point B splits into two points, C and D. The circumference of the sector is $2\pi r$, because it is the same as the circumference of the base. The radius of the sector is $\sqrt{r^2 + h^2}$. Hence the central angle of the sector is $\frac{2\pi r}{\sqrt{r^2+h^2}}$. Therefore, the area of the sector is

$$\pi(r^2 + h^2)\left(\frac{2\pi r}{2\pi\sqrt{r^2 + h^2}}\right),$$

which simplifies to $\pi r\sqrt{r^2 + h^2}$. Finally, the total surface area is

$$\pi r(r + \sqrt{r^2 + h^2}).$$

0.3.4 Sphere

The surface area of a sphere with radius r is equal to $4\pi r^2$. This formula is obtained by approximating the surface by covering it with small rectangles.

> **Long term challenge** : Can you prove this? If not, can you find an intuitive explanation?