Selected Problems and Theorems in Elementary Mathematics

Arithmetic and Algebra

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ИЗДАТЕЛЬСТВО
МИР

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Preface

The present book contains 350 problems selected from the material of the mathematical olympiads and school mathematics hobby groups in Moscow. About 15 problems have been taken from the manuscript of the late D. O. Shklyarsky (1918-1942), one of the founders of the mathematics hobby group for pupils at the State University of Moscow.

It is a great pleasure to thank I. Bernstein, N. Vasilyev, G. Galperin, Yu. Ionin, A. Leman, A. Savin, A. Tolpygo, A. Toom, V. Gutenmacher, L. Makar-Limanov and L. I. Golovina for valuable advice and help in the preparation of the book.

I.M. Yaglom

Contents

Instructions

This book contains the conditions of problems, the answers and hints to them and the solutions of the problems. The conditions of the most difficult problems are marked by stars.

We recommend the reader to start with trying to solve without assistance the problem he is interested in. In case this attempt fails he can read the hint or the answer to the problem, which may facilitate the solution. Finally, if this does not help, the solution of the problem given in the book should be studied. However, for the starred problems it may turn out to be appropriate to begin with reading the hints or the answers before proceeding to solve the problems.

Most of the problems in the book are independent of one another except those in the last two sections ("Complex Numbers" and "Several Problems in Number Theory") where the problems are more closely interrelated.

It is advisable to choose a definite section of the book and to spend some time on solving the problems of that section. Only after that (this does not of course mean that all the problems or most of the problems must necessarily be solved) should the reader pass to another section and so on. However, the order in which the sections are arranged in the book may not be followed. The solutions of some problems include indications concerning possible generalizations of the conditions of the problems. The reader is advised to think of similar generalizations for other problems; it is also interesting to try to state new problems akin to those collected in this book.

Problems

1. Introductory Problems

Most of the problems collected in this section are exercises meant for logical training and they are not connected with any definite division of mathematics. Some of these problems are purely arithmetical (for instance, see Problems 25-29) while some others can be associated with the *graph theory*. By a *graph* is meant a system of points (see Fig. 1*a*) some of which are connected by lines. Sometimes certain directions of motion are indicated by arrows on some (or all) of these lines. Then we speak of a *directed* graph (see Fig. 1*b*). For instance, those of the problems below which are related to transportation systems can be stated in terms of the graph theory (and systems of roads with one-way traffic should naturally be represented by means of directed graphs). Similarly, a group of people some of whom are acquainted with one another can also be represented as a system of points among which those representing the people acquainted with one another are connected by lines *.

Most of the problems of this section do not require any special knowledge of mathematics and therefore their solutions can easily be understood by junior pupils. However, the solutions of some of the problems are based on the *method of mathematical induction* with which usually only senior pupils are familiar. For the solution of some other problems *Dirichlet's principle* ** can be of use; conditionally, this principle is stated as follows: *if there are seven rabbits and five cages* (*or, generally, m rabbits and n cages where $n < m$*) *and if it is required to put the rabbits in the cages then it is necessary to put two* (*or more*) *rabbits in at least one cage.*

1. Two hundred soldiers form a rectangular array with ten soldiers in each line and 20 soldiers in each file. From each line the smallest soldier is chosen, after which among the 20 soldiers thus taken the tallest one is chosen. Then from each file of the same array of 200 soldiers the tallest soldier is chosen, after which

* In the problems of the present section an acquaintance relation is always assumed to be *symmetric* in the sense that if a person A is said to be acquainted with a person B then it is automatically meant that B is acquainted with A. If this convention is not introduced then a system of acquaintance relations should be represented by a *directed* graph.

** Peter Gustav Lejeune Dirichlet (1805-1859), a distinguished German mathematician.

among the 10 soldiers thus taken the smallest one is chosen. Which of the two soldiers, that is the smallest among the tallest soldiers or the tallest among the smallest soldiers (provided that these are different persons), is taller?

2. Each of the people who has ever lived on the Earth has shaken hands with a number of other people. Prove that the number of people each of whom has shaken hands an odd number of times is even.

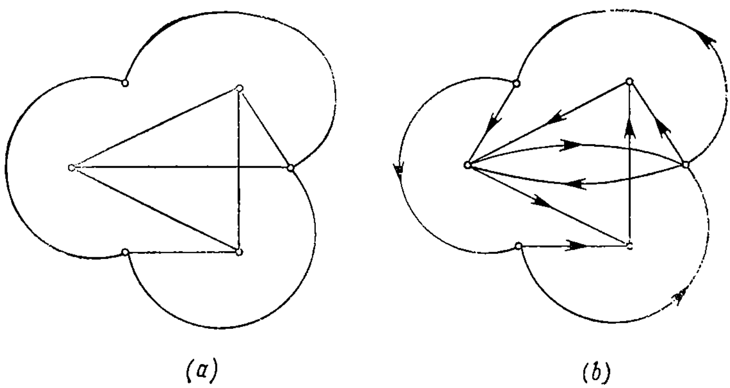

Fig. 1

3. Prove that among any six people there are three people pairwise acquainted or three people pairwise not acquainted.

4. Several people take part in a meeting (it is of course meant that the number of the people exceeds one because, if otherwise, it would be senseless to speak of a "meeting"!).

(a) Is it possible that among them there are not two persons who are acquainted with the same number of people present at the meeting?

(b) Prove that there can be the case when for any number of the participants of the meeting there are not three people each of whom is acquainted with the same number of people present at the meeting.

5. $2n$ people take part in a meeting and each of them is acquainted with not less than n people present. Prove that among these people there are four persons who can be seated at a round table so that each of them is acquainted with the neighbours sitting on his left and on his right.

6. A number of scientists took part in a congress. Some of them had been acquainted with some other participants of the congress before and some were not. It turned out that among the scientists there were not two persons who were acquainted with the same number of participants and who had a mutual acquaintance. Prove that among the scientists who attended the congress there was a person who was acquainted with only one of the participants.

7. At a congress there are 1000 delegates from various countries. It is known that every three delegates can speak with one another without the help of the rest (but it may happen that one of the three persons has to serve as an interpreter for the other two). Prove that all the participants of the congress can be put up at a hotel with double rooms so that in each room there are two delegates who can speak with each other.

8. Seventeen scientists take part in an international conference. There are three languages such that each of the 17 scientists knows at least one of them. It is known that every two participants of the conference can speak with each other in at least one of the three languages. Prove that among the participants of the conference there are three persons who can speak with one another in one and the same language.

9. There are n people at a meeting. It is known that every two of the participants of the meeting who are acquainted with each other have no mutual acquaintances and that every two participants who are not acquainted with each other have exactly two mutual acquaintances.

(a) Prove that all the participants have the same number of acquaintances.

(b) For what n can the conditions of the problem be fulfilled?

10. In the town of "Manifold" there are 10 000 inhabitants and every two of them are either friends or enemies. Every day not more than one of the inhabitants of the town can quarrel with all his friends and, simultaneously, make friends with all his enemies; besides, any three inhabitants can make friends with one another. Prove that in a number of days all the inhabitants without exception can make friends with one another. What is the least number of days sufficient for it?

11.* In the State of Oz there are several castles from each of which three roads start. A knight-errant leaves his ancestral castle to travel in the country. The knight is fond of variety and therefore when he arrives at a castle he always turns to the left if he turned to the right the previous time and turns to the right if he turned to the left the previous time. (When going past the first castle on his way the knight may turn in any direction.) Prove that eventually the knight will return to his own castle.

12.* $2n$ Knights of the Round Table gathered at King Arthur's court, each of them having not more than $n-1$ enemies among the knights present. Prove that Merlin (King Arthur's Counsellor) can seat the knights at the round table so that none of them sits next to his enemy.

13. **(a)** Among 80 given coins one coin is known to be false (it is also known that the false coin is lighter than a genuine coin; all the genuine coins are of the same weight). It is required to detect the false coin by means of four weighings using a beam balance without weights.

(b) It is known that among n given coins there is a false one which is lighter than a genuine coin; all the genuine coins are of the same weight. What is the least number k such that it is always possible to detect the false coin by means of k weighings using a beam balance without weights?

14. There are 20 metal cubes of the same size and look some of which are made of aluminium while the others are made of duralumin, the latter being heavier. How can we determine the number of the cubes made of duralumin with the aid of not more than 11 weighings using a beam balance without weights?

Remark. In this problem we assume that it is possible that all the cubes are made of aluminium and that they cannot be all made of duralumin (because without this assumption it would be impossible to find whether the cubes are made of aluminium or of duralumin in case all the cubes turn out to be of the same weight).

15.* There is a false coin among 12 given coins. It is known that the false coin differs in its weight from a genuine coin but it is unkown whether it is lighter or heavier. All the genuine coins are of the same weight. It is required to detect the false coin with the aid of three weighings using a beam balance without weights and, simultaneously, to find whether that coin is lighter or heavier than the other coins.

Remark. Under the conditions given in Problem 15 it is possible, using three weighings, to detect the false coin not only among 12 but also among 13 given coins; however, in the latter case it is impossble to find whether the false coin is lighter or heavier than a genuine coin. It turns out that 14 coins need four weighings.

It can also be proved (although the proof is rather intricate!) that if we are given an arbitrary N of coins one of which is false and differs in its weight from a genuine coin (all the genuine coins are of the same weight) then *the least number k of weighings with the aid of a beam balance without weights making it possible to detect the false coin and simultaneously, to find whether it is lighter or heavier than a genuine coin is equal to* $\log_3(2N+3)$ *in case the number* $2N+3$ *is equal to an integral power of the number* 3 *and is equal to* $[\log_3(2N+3)+1]$ *in case* $2N+3$ *is not equal to an integral power of* 3 (that is in case the number $\log_3(2N+3)$ is not integral). Here the square brackets denote the *integral part* of a number (see page 36). For $N=12$ this general statement implies that $k=3$. For the general case of N coins it would also be interesting to determine the least number k_1 of weighings making it possible to detect the false coin without finding whether it is lighter or heavier than a genuine coin (for $N=12$ or $N=13$ we have $k_1=3$ while for $N=14$ we have $k_1=4$; so far as we know the general expression $k_1=k_1\ (N)$ has not yet been determined).

16. **(a)** Once a man entered an inn. He had no money but he had a silver chain consisting of seven links. He was put up at the inn and it was agreed that every day he would give the innkeeper one of the links of the chain. What is the least number of the links of the chain that must be cut so that the man can pay the innkeeper for seven days (if necessary, the man can take back from the innkeeper some of the links he has already given to him and give him some other links in exchange)?

(b) A chain consists of 2000 links. What is the least number of the links of the chain that should be cut so that it is possible to take any number of links ranging from 1 to 2000 by using the parts of the chain thus obtained?

17. In the town of Liss all the underground stations are connected so that it is possible to go from any station to any other (if necessary, the passengers are allowed to change trains). Prove that in these conditions there is an underground station such that when it is closed (the trains are not allowed to go past the station which is closed) it is still possible to go from any of the remaining stations to any other.

18*. There was two-way traffic in all the streets of the town of Zurbagan. When it was necessary to have all the roads repaired the municipal authorities had to introduce temporarily one-way traffic in some of the streets, two-way traffic remaining in the rest of the streets. After part of the streets were repaired two-way traffic was restored in them and in the others one-ray traffic was introduced. During both periods of the repairs it was possible to go from any place of Zurbagan to any other place. Prove that one-way traffic can be introduced in all the streets of the town in such a way that it is possible to go from any place to any other.

19*. There are n towns in the state of Dolphinia every two of which are connected by a road, the traffic in the roads being one-way. Prove that if $n \neq 2$ or $n \neq 4$ then the direction of the movement along the roads can be chosen so that one can go from any town to any other town without going through more than one town. Also prove that for the case $n = 2$ or $n = 4$ such organization of traffic is impossible.

20*. In the state of Shvambrania there are 100 towns. It is known that if two towns A and B have no direct telephone communication then there are air routes from A to B and from B to A and that if there is direct telephone communication between A and B then there are no such routes. It is also known that any two towns in Shvambrania can have telephone communication (possibly with the aid of several intermediate telephone exchanges) and that it is possible to go by air from any town to any other town (possibly with several landings). Prove that there are four towns in Shvambrania such that there can be telephone communication between any two of them and one can fly from any of these towns to any other using, if necessary, only two of these four towns as intermediate points.

21. Can a knight move from the left lower corner of an ordinary chess-board to the right upper corner passing through each of the squares of the chess-board exactly once?

22. *A king's suicide problem.* On a chess-board of 1000×1000 squares there are 499 black rooks and a white king. Prove that for arbitrary initial positions of all these chessmen and for an arbitrary strategy of the black the king can "play at give-away", that is arrive in several moves at a square where it must be taken by one of the rooks. (The chessmen on the chess-board are supposed to move according to the ordinary rules.)

23. Twelve squares are arranged in a circular order and four neighbouring squares are occupied by four counters of different colour: red, yellow, green and blue.

Any counter can be moved from the square it occupies across any four squares to the fifth one (provided that the latter is not occupied) in any of the two possible directions. After a number of such moves the counters may again occupy the four initial squares. What permutations of the counters can we have in this case?

24. The students admitted to a university include exactly 50 speaking English, exactly 50 speaking French and exactly 50 speaking German. Of course, some of the students may speak two or three of the languages and therefore, in the general case, the total number of the students (each of whom speaks at least one of the languages) may be less than $3 \cdot 50 = 150$. Prove that all the students can be divided into 5 groups (generally consisting of a different number of students) so that each group contains exactly 10 people speaking English, exactly 10 people speaking French and exactly 10 people speaking German.

25. **(a)** Twenty athletes took part in a contest, and there were 9 referees. According to his judgement on the achievements of the athletes every referee made a list in which he arranged the athletes from the 1st to the 20th place. It turned out that there was no considerable difference in the judgment of all the referees: the places which each of the athletes was given by any two of the referees differed by not more than three. The final distribution of the places was done by determining the "average place" of every athlete, that is by dividing by 9 the sum of the places he was given by all the nine referees. What is the greatest possible value of the "average place" of the best of the 20 athletes?

(b) The Tennis Federation gave qualification numbers to all the tennis-players of the country: the best player received the 1st number, the next received the 2nd number and so on. It is known that in a game of any two of the tennis-players whose numbers differ by more than 2 the one having a smaller number always wins.

In the Olympic games the 1024 tennis-players of the country take part (this means that after every round of the contest all the losers leave and the rest of the participants are divided into contesting pairs in a random way and then take part in the next round). What is the greatest value of the qualification number the winner of such games can have?

*26**. The Games lasted n days and N sets of medals were awarded to the winners during the Games: one set of medals and 1/7 of the remaining medals on the 1st day, 2 sets of medals and 1/7 of the remaining part on the 2nd day, ..., $(n-1)$ sets of medals and 1/7 of the rest on the $(n-1)$th day (the last but one day) and, finally, all the n remaining sets of medals on the last day. How many days did the Games last and how many sets of medals were awarded to the winners?

27. There were five friends one of whom had a monkey. Once they bought a bag of nuts and decided to share the nuts among themselves the next morning. At night one of them woke up. He divided the nuts into five equal parts, found that one extra nut remained after the division, gave it to the monkey, ate his part of the nuts and fell asleep again. After that another owner of the nuts woke up. He did not know that some of the nuts had been taken and therefore he divided all the nuts remaining in the bag into five equal parts. He also found that there remained one nut after the division which he gave to the monkey. He ate one of these five parts and fell asleep. Then the three remaining friends performed, in succession, the same operations, that is, each of them divided the rest of the nuts into five parts not knowing what his friends had done, found that there remained one nut after the division, gave it to the monkey and ate one of the five parts. Finally, in the morning all the five friends divided the remaining nuts into five parts, saw that there remained one nut after the sharing and gave it to the monkey. It is required to determine the least possible number of the nuts in the bag for such a sharing to be possible.

28. Two brothers had a flock of sheep. They sold the flock and got as many rubles for every sheep as was the number of the sheep in the flock. The money was shared in the following way: first the elder brother took ten rubles from the cash, then the younger brother took ten rubles, after which the elder brother took ten rubles again and so on. Finally, it turned out that at the last stage when it was the younger brother's turn to take money there remained less than ten rubles. Therefore the younger brother took the rest of the money and the elder brother gave him his knife for the sharing to be fair. How much did the knife cost?

29. **(a)** Which of the two days, Saturday or Sunday, happens to be more frequently a New Year's Day?

(b) What day of the week happens to be most frequently the 30th day of a month?

2. Permutation of Digits in a Number

30. A whole number decreases an integral number of times when its last digit is deleted. Find all such numbers.

31. **(a)** Find all whole numbers which begin with the digit 6 and decrease 25 times when this digit is deleted.

(b) Prove that there is no whole number which decreases 35 times when its initial digit is deleted.

32.* A whole number decreases 9 times when one of its digits is deleted, and the resultant number is divisible by 9.

(a) Prove that in order to divide the resultant number by 9 it is also sufficient to delete one of its digits.

(b) Find all the whole numbers satisfying the condition of the problem.

33. **(a)** Find all whole numbers which decrease an integral number of times when their third digits are deleted.

(b)* Find all whole numbers which decrease an integral number of times when their second digits are deleted.

34. **(a)** Find the least whole number which begins with the digit 1 and increases 3 times when this digit is carried to the end of the number. Find all the numbers possessing this property.

(b) What digits can stand at the beginning of the whole numbers which increase three times when these initial digits are carried to the end of the numbers? Find all such numbers.

35. Find the least natural number whose last digit is 6 such that it increases 4 times when this last digit is carried to the beginning of the number.

36. Prove that there are no positive integral numbers which increase 5 or 6 or 8 times when their initial digits are carried to the end of the numbers.

37. Prove that there are no positive integral numbers which increase twice when their initial digits are carried to the end of the numbers.

38. **(a)** Prove that there are no positive integral numbers which increase 7 or 9 times when their initial digits are carried to the end.

(b) Prove that there are no positive integral numbers which increase 4 times when their initial digits are carried to the end of the numbers.

39. Find the least whole number whose initial digit is 7 which decreases 3 times when this digit is carried to the end of the number. Find all such numbers.

40. (a) Prove that a positive integer cannot be 2, 3, 5, 6, 7 or 8 times as small as its "reversion", that is as the number consisting of the same digits written in the reverse order.

(b)*. Find all positive integers which are 4 or 9 times as small as their reversions.

41. (a) Find a 6-digit number which increases 6 times when its three last digits are carried to the beginning of the number without their order being changed.

(b) Prove that there exists no 8-digit number which increases 6 times when its last four digits are carried to the beginning of the number with the preservation of their order.

42. Find a 6-digit number whose products by 2, 3, 4, 5 and 6 are written with the aid of the same digits as the original number but in some other order.

43. A whole number is equal to the arithmetic mean of all the numbers obtained from the given number with the aid of all the possible permutations of its digits (including, of course, the "identity permutation" under which all the digits retain their places). Find all whole numbers possessing this property.

44. Let A be a positive integer and A' be a number written with the aid of the same digits which are arranged in some other order. Prove that if $A + A' = 10^{10}$ then A is divisible by 10.

45. Let M be a 17-digit number and N be the number obtained from M by writing the same digits in the reverse order. Prove that at least one digit in the decimal representation of the number $M + N$ is even.

3. Problems in Divisibility of Numbers

Most of the topics whose study is started in this section are related to "higher arithmetic", that is to *number theory*. The study is in some way continued in the following sections and first of all in Secs. 4, 5 and 11.

46. Prove that for any integer n

(a) $n^3 - n$ is divisible by 3;

(b) $n^5 - n$ is divisible by 5;

(c) $n^7 - n$ is divisible by 7;

(d) $n^{11} - n$ is divisible by 11;

(e) $n^{13} - n$ is divisible by 13.

Remark. Note that n^9-n must not necessarily be divisible by 9 (for instance, $2^9-2=510$ is not divisible by 9).

Problems (a)-(e) deal with special cases of a more general theorem; see Problem 340 (page 67).

47. Prove that for any integer n

(a) $3^{6n}-2^{6n}$ is divisible by 35 (here $n \geqslant 0$);

(b) n^5-5n^3+4n is divisible by 120;

(c) n^2+3n+5 is not divisible by 121.

48. Prove that for any integers m and n

(a)* $mn(m^{60}-n^{60})$ is divisible by 56 786 730;

(b) $m^5+3m^4n-5m^3n^2-15m^2n^3+4mn^4+12n^5$ is not equal to 33.

49. For what positive integers n the number $20^n+16^n-3^n-1$ is divisible by 323?

50. Is there a natural number n such that n^2+n+1 is divisible by 1955?

51. What number can be obtained in the remainder when the hundredth power of a whole number is divided by 125?

52. Prove that if a whole number N is relatively prime to 10 then the 101th power of the number N has the same last three digits as N (for instance, the last three digits of 1233^{101} are 233 and those of 37^{101} are 037).

53. Find a three-digit number which, when raised to any integral power, gives a number whose last three digits form the original number.

54*. Let N be an even number not divisible by 10. What digit is in the tens place of the number N^{20}? What digit is in the hundreds place of the number N^{200}?

55. Prove that a sum of the form

$$1^k+2^k+3^k+\ldots+n^k$$

where n is an arbitrary positive integer and k is odd, is divisible by $1+2+3+\ldots+n$.

56. Derive the test for divisibility of whole numbers by 11.

57. The number 123456789(10)(11)(12)(13)(14) is written in the number system to base 15, that is this number is equal to $(14)+(13)\cdot 15+(12)\cdot 15^2+(11)\cdot 15^3+\ldots+2\cdot 15^{12}+15^{13}$. What number is obtained in the remainder when the given number is divided by 7?

58. Let us consider all numbers K such that if a number N is divisible by K then every number obtained from the number N by any permutation of its digits is also divisible by K. Prove that K can only be equal to 1, 3 and 9. (For $K=1$ the indicated property is quite obvious and for $K=3$ and $K=9$ it follows from the well-known tests for divisibility by 3 and by 9.)

59. Find the least whole number whose decimal representation consists only of 1's which is exactly divisible by the number $\underbrace{333\ldots33}_{100\text{ threes}}$.

60. Let a be the last digit of a whole number $N = 2^k$ and A be the number obtained by deleting this last digit in N. Prove that for all $k > 3$ the number aA is divisible by 6.

61. Prove that the expression $27\,195^8 - 10\,887^8 + 10\,152^8$ is exactly divisible by 26 460.

62. Prove that $11^{10} - 1$ is divisible by 100.

63. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

64. Prove that any number composed of 3^n similar digits is divisible by 3^n (for example, the number 222 is divisible by 3, the number 777 777 777 is divisible by 9 and so on).

65. Find the remainder which is obtained when the number

$$10^{10} + 10^{(10^2)} + \ldots + 10^{(10^{10})}$$

is divided by 7.

66. **(a)** Find the last digit of the number $9^{(9^9)}$ and of the number $2^{(3^4)}$.

(b) Find the last two digits of the numbers 2^{999} and 3^{999}.

(c)* Find the last two digits of the number $14^{(14^{14})}$.

67. Prove that

(a) the decimal representations of the numbers 9^{9^9} and $9^{9^{9^9}}$ (where, for instance, 9^{9^9} means $9^{(9^9)}$) have the same last two digits.

(b)* the decimal representations of the numbers 7^{7^7} and $7^{7^{7^{7^7}}}$ have the same last six digits.

68. **(a)** What is the last digit of the decimal representation of the number

$$(\ldots(((7^7)^7)^7)^{\cdots 7})$$

where the raising to the power 7 is repeated 1000 times? What are the last two digits of that number?

(b) What is the last digit in the decimal representation of the number

$$7^{\left(7^{\cdot^{\cdot^{\cdot^{(7^{(7^7)})}}}}\ldots\right)}$$

which is written with the aid of 1001 sevens as in Problem (a) but the order of the raising to the power is changed? What are the last two digits of this number?

*69**. Let us consider the number

$$N = 9^{\left(9^{\cdot^{\cdot^{\cdot^{\left(9^{\left(9^{9}\right)}\right)}}}}\ \ldots\right)}$$

written with the aid of 1001 nines by analogy with the number in Problem 68 (b). Find the last five digits of this number.

70. For what natural numbers n is the sum $5^n + n^5$ divisible by 13? What is the least number n satisfying this condition?

71. Find the last two digits of the number

$$n^a + (n+1)^a + (n+2)^a + \ldots + (n+99)^a$$

where n is an arbitrary nonnegative integer and

(a) $a = 4$;

(b) $a = 8$.

*72**. Find the last 1000 digits of the number

$$N = 1 + 50 + 50^2 + 50^3 + \ldots + 50^{999}$$

73. A natural number M is divisible by 7; prove that if the number of the digits in the decimal representation of the number M is divisible by 6 then the number N obtained by carrying the last digit of M to its beginning is also divisible by 7.

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74. How many noughts stand at the end of the product of all whole numbers from 1 to 100 inclusive?

We shall use the notation

$$1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1) \cdot n = n!$$

($n!$ is called *factorial* n). The problem can briefly be stated as follows: how many noughts are there at the end of the number 100!?

75. **(a)** Prove that a product of n consecutive whole numbers is divisible by $n!$.

(b) Prove that a fraction of the form $\frac{n!}{a!\,b!\ldots k!}$ is equal to a whole number provided that $a + b + \ldots + k \leqslant n$.

(c) Prove that $(n!)!$ is divisible by $n!^{(n-1)!}$.

(d)* Prove that a product of n whole numbers forming an arithmetic progression whose common difference is relatively prime to $n!$ is divisible by $n!$.

Remark. Problem 75 (d) is a generalization of Problem 75 (a).

76. Is the number of combinations of 1000 things taken 500 at a time divisible by 7?

77. **(a)** Find all those numbers n lying between 1 and 100 for which $(n-1)!$ is not divisible by n.

(b) Find all those numbers n lying between 1 and 100 for which $(n-1)!$ is not divisible by n^2.

78.* Find all whole numbers n divisible by all whole numbers not exceeding $\sqrt{n}$.

79. (a) Prove that a sum of squares of five consecutive whole numbers cannot be a perfect square of a whole number.

(b) Prove that a sum of powers of three consecutive whole numbers with equal even exponents cannot be equal to an even power of a whole number.

(c) Prove that a sum of powers of nine consecutive whole numbers with equal even exponents cannot be equal to any power (of course, with an exponent exceeding 1) of a whole number.

80. (a) Let A and B be two different seven-digit numbers each of which is composed of all the digits from 1 to 7. Prove that A is not divisible by B.

(b) Using all the digits from 1 to 9 compose three 3-digit numbers which are in the ratio 1 : 2 : 3.

81. A square of a whole number has four equal digits at its end. What are these digits?

82. Prove that if the lengths of two sides of a rectangle and of its diagonal are expressed by whole numbers then the area of the rectangle is divisible by 12.

83. Prove that if the coefficients of a quadratic equation

$$ax^2 + bx + c = 0$$

are odd integers then the roots of the equation cannot be rational numbers.

84. Prove that if the sum of fractions

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$$

where n is a whole number, is written in decimal notation then the resultant expression is a mixed periodic decimal.

85. Prove that the expressions

(a) $M = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$;

(b) $N = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+m}$;

(c) $K = \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}$

where n and m are positive integers, cannot be equal to whole numbers.

86. (a) Prove that a fraction of the form $\frac{a^3 + 2a}{a^4 + 3a^2 + 1}$ cannot be reduced by a factor for any integral value of a.

(b) Find all (natural) numbers by which a fraction $\frac{5n+6}{8n+7}$ can be reduced for an integral n.

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87. 1953 digits are written in a circular order. Prove that if the 1953-digit numbers obtained when we read these digits in clockwise direction beginning with one of the digits is divisible by 27 then if we read these digits in the same direction beginning with any other digit the new 1953-digit number is also divisible by 27.

88. Prove that there exists a number divisible by 5^{1000} whose decimal representation involves no noughts.

89. Prove that all numbers of the form 10 001; 100 010 001; 1 000 100 010 001, ... are composite.

90. Prove that any two numbers in the sequence

$$2+1, \quad 2^2+1, \quad 2^4+1, \quad 2^8+1, \quad 2^{16}+1, \ldots, \quad 2^{2^n}+1, \ldots$$

are relatively prime.

Remark. In particular, the result of this problem implies that there are infinitely many prime numbers (in this connection also see Problems 234 and 349). Indeed, if the set of the prime numbers were finite there could not exist infinitely many numbers among which any two numbers are relatively prime.

91. Prove that if one of the numbers $2^n - 1$ and $2^n + 1$ where $n > 2$ is prime then the other number is composite (for $n = 2$ both $2^n - 1 = 3$ and $2^n + 1 = 5$ are prime numbers).

92. **(a)** Prove that if p and $8p - 1$ are prime numbers then $8p + 1$ is a composite number.

(b) Prove that if p and $8p^2 + 1$ are prime numbers then $8p^2 - 1$ is also a prime number.

93. Prove that when any prime number different from 2 and 3 is divided by 12 we obtain 1 in the remainder.

94. Prove that if three prime numbers exceeding the number 3 form an arithmetic progression then the common difference of the progression is divisible by 6.

95*. **(a)** Ten prime numbers each of which is less than 3000 form an arithmetic progression. Find these numbers.

(b) Prove that there are not 11 prime numbers each of which is less than 20 000 such that they form an arithmetic progression.

96. **(a)** Prove that from any five consecutive whole numbers it is always possible to choose a number which is relatively prime to the other four numbers.

(b) Prove that, given 16 consecutive whole numbers, it is always possible to choose a number from them which is relatively prime to the other 15 numbers.

4. Miscellaneous Problems in Arithmetic

97. A number A is written in decimal number system with the aid of 666 threes and a number B with the aid of 666 sixes. Of what digits does the decimal representation of the product $A \cdot B$ consist?

98. A decimal representation of a number A consists of 1001 sevens. Find the quotient and the remainder resulting from the division of A by the number 1001.

99. Find the least square (of a whole number) whose decimal representation starts with six 2's.

100. Are there whole numbers m and n such that $m^2 = n^2 + 1954$?

101. Add three digits to 523 so that the resultant six-digit number is divisible by 7, by 8 and by 9.

102. Find a four-digit number whose division by 131 leaves a remainder of 112 and whose division by 132 leaves a remainder of 98.

103. (a) Prove that the sum of all n-digit numbers ($n > 2$) is equal to $494\underbrace{99\ldots9}_{(n-3)\text{ times}}55\underbrace{00\ldots0}_{(n-2)\text{ times}}$ (for instance, the sum of all three-digit numbers is equal to 494 550 and the sum of all six-digit numbers is equal to 494 999 550 000).

(b) Find the sum of all even four-digit numbers which can be written with the aid of the digits 0, 1, 2, 3, 4 and 5 (it is allowable to repeat any digit in a number).

104. How many digits and what digits are needed to write all whole numbers from 1 to 100 000 000 inclusive?

105. Suppose that all whole numbers are consecutively written down from left to right. Find the 206 788th digit in this infinite sequence.

106. Let us consider an infinite decimal of the form 0.1234567891011121314 ... where all the whole numbers are consecutively written after the decimal point. Is this decimal periodic?

107. Each of the whole numbers from 1 to 1 000 000 000 inclusive is replaced by the sum of the digits forming the number (of course, under this operation 1-digit numbers do not change whereas all the other numbers decrease). Then each of the resultant numbers is again replaced by the sum of its digits, and the operation is performed repeatedly until we obtain a sequence of 1-digit numbers containing 1 000 000 000 members. Is the number of 1's in this sequence greater than the number of 2's or not?

108. (a) A decimal representation of a whole number involves only a number of sixes and a number of noughts. Can this number be a perfect square?

(b) Answer the same question for a whole number in whose decimal representation the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are present, each of the digits is used only once, and the digit 5 stands at the end of the number.

109. Each of the five digit numbers from 11 111 to 99 999 inclusive is written on a separate card (the number of these cards is obviously equal to 88 889). Then the cards are arranged in an arbitrary manner to form a chain. Prove that the 444 445-digit number obtained in this way ($444\,445 = 88\,889 \cdot 5$) is not equal to a power of two.

110. In the decimal representation of a 10-digit number the initial digit is equal to the number of noughts in the representation, the next digit is equal to the number of ones and so on (accordingly, the last digit is equal to the number of nines in the representation). Find all such 10-digit numbers.

111. By what factor should the number 999 999 999 be multiplied in order to obtain a number consisting only of ones?

112. Let A be a natural number. Prove that there exist infinitely many (natural) numbers N whose decimal representations involve only the digits 1, 2, ..., 9 (and do not involve noughts!) such that the sums of the digits in the decimal representations of the numbers N and AN are equal *.

113. Let $a_1, a_2, \ldots$ be all nonnegative integers with not more than $n (n \geqslant 2)$ decimal places for which the sums of their digits are even and let $b_1, b_2, \ldots$ be all non-negative integers with not more than n decimal places for which the sums of their digits are odd. Prove that

$$a_1^m + a_2^m + \ldots = b_1^m + b_2^m + \ldots$$

for all (natural) $m < n$. Does this assertion remain true for $m \geqslant n$?

114. In the triangular number array

$$\begin{array}{c} 1 \\ 1\;1\;1 \\ 1\;2\;3\;2\;1 \\ 1\;3\;6\;7\;6\;3\;1 \\ \cdots\cdots\cdots \end{array}$$

* The stipulation that the decimal representations of the numbers N do not involve noughts is made because if we write any number of noughts at the end of the representation of N the sums of the digits in the (new) number N and in the number AN do not of course change, and therefore without this stipulation the existence of only one number N satisfying the condition of the problem would automatically imply the existence of an *infinitude* of such numbers.

each of the numbers is equal to the sum of the three numbers written in the preceding horizontal row above the given number and above the numbers standing on the right and on the left of this number; in case one of the two such numbers is absent in the preceding row they are replaced by zeros.

Prove that, beginning with the third row, there is an even number in every row.

115. Consider the triangular number array

```
0  1  2  3  . . . . . . . . . . . . 1956   1957   1958
  1  3  5 . . . . . . . . . . . .     3913   3915
    4  8 . . . . . . . . . . . . . .     7828
          . . . . . . . . . . . . . . .
              . . . . . . . . . .
```

in which every number except those in the upper horizontal row is equal to the sum of the two numbers standing above this number in the preceding row. Prove that the last number standing in the lowermost row is divisible by 1958.

116. The distance between two stations A and B is equal to 999 km. Kilometre poles along the railway connecting A and B show the distances from the poles to A and to B. They read thus:

$$0|999;\quad 1|998;\quad 2|997;\ \ldots;\ 999|0$$

How many of these poles are such that there are only two different digits on them?

117. A boy passing by the cinema on a bus could notice only the hours (but not the minutes!) when four (of the eight) shows began:

1st show —12 (hours) . . . (minutes)
2nd show —13 (hours) . . . (minutes)
. . . .
7th show —23 (hours) . . . (minutes)
8th show —24 (hours) . . . (minutes)

It is required to restore from these data the exact time of the beginning of all the shows (it is implied that the duration of all the eight shows is the same).

118. A highway with round-the-clock bus service crosses a railway. Every hour two trains run along the railway and approach the level crossing exactly at n hours and at n hours 38 minutes respectively where n assumes the values from 0 to 23. When a train passes the crossing the lifting gate stops the road traffic for 5 minutes. Is it possible to work out a timetable for the buses so that they go with an interval of T minutes and so that no bus stops at the crossing? For what intervals T between the buses not exceeding half an hour is it possible to schedule the bus service in the required manner?

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119. Find the greatest possible value of the ratio of a three-digit number to the sum of its digits.

120. Delete 100 digits in the number

$$12345678910111213\ldots979899100$$

so that the resultant number has

(a) the greatest possible value;

(b) the least possible value.

121. Using all the digits from 1 to 9 compose three 3-digit numbers so that their product has

(a) the least possible value;

(b) the greatest possible value.

122. A sum of several consecutive positive integers is equal to 1000. Find these integers.

123. (a) Prove that every whole number which is not equal to a power of two can be represented in the form of a sum of at least two consecutive positive integers and that for the powers of two such a representation is impossible.

(b) Prove that every odd composite number can be represented as a sum of at least two consecutive odd numbers and that no prime number can be represented in that way. What even numbers can be represented in the form of a sum of several consecutive odd numbers?

(c) Prove that every power of a positive integer n (with the exponent greater than 1) can be represented as a sum of n consecutive odd numbers.

124. Prove that every sum of 1 and a product of four consecutive whole numbers is a perfect square.

125. Let us consider a collection of $4n$ positive numbers such that its any four pairwise different members can be arranged as a geometric progression. Prove that there are n equal numbers among the given collection of numbers.

126. There are 27 weights of magnitudes $1^2, 2^2, 3^2, \ldots, 27^2$ respectively. It is required to divide them into three groups of equal weight.

127. There are 13 weights, each weighing an integral number of grams. It is known that any 12 of the weights can be divided into two groups of 6 weights balancing each other when put on the scales. Prove that all the weights are identical.

128*. Four (arbitrary) numbers a, b, c and d are written in a line. Another 4-tuple consisting of the numbers $a_1 = ab$, $b_1 = bc$, $c_1 = cd$ and $d_1 = da$ is written under these numbers. Then under the numbers a_1, b_1, c_1 and d_1 a new 4-tuple $a_2 = a_1b_1$, $b_2 = b_1c_1$, $c_2 = c_2d_1$ and $d_2 = d_1a_1$ is written and so on. Prove that all the 4-tuples thus formed either are pairwise different or, beginning with one of them, become identical.

129. There is an arbitrary set of N numbers $a_1, a_2, \ldots, a_N$ (where N is an exact power of two: $N = 2^k$) each of which is equal to $+1$ or to -1. Starting with this set, a new number set is formed according to the formulas $a_1' = a_1a_2$, $a_2' = a_2a_3$, $\ldots$, $a_{N-1}' = a_{N-1}a_N$, $a_N' = a_Na_1$, each of the new numbers being again equal to $+1$ or to -1. Then, using the numbers $a_1', a_2', \ldots, a_N'$, a new N-tuple of numbers $a_1'', a_2'', \ldots, a_N''$ is formed in accordance with the above rule, and so on. Prove that proceeding in this way we eventually arrive at an N-tuple consisting only of the numbers $+1$.

130*. Let $a_1, a_2, \ldots, a_n$ where $n > 2$ be integers. Using these numbers a new sequence consisting of the numbers $a_1' = \frac{a_1 + a_2}{2}$, $a_2' = \frac{a_2 + a_3}{2}$, $\ldots$, $a_{n-1}' = \frac{a_{n-1} + a_n}{2}$, $a_n' = \frac{a_n + a_1}{2}$ is formed. Then, proceeding from the numbers $a_1', a_2', \ldots, a_n'$, new numbers a_1'', a_2'', $\ldots$, a_n'' are formed in accordance with the same rule (that is $a_1'' = \frac{a_1' + a_2'}{2}$, a_2'', $\ldots$, a_n'', etc.), and so on. Prove that if all the numbers thus obtained are integers then $a_1 = a_2 = \ldots = a_n$.

131. Let $x = 1$ and let y and z be arbitrary numbers. We shall denote the absolute values $|x - y|$, $|y - z|$ and $|z - x|$ of the pairwise differences of the three original numbers as x_1, y_1 and z_1 respectively. Similarly, we shall denote the absolute values of the pairwise differences of the numbers x_1, y_1, and z_1, that is the quantities $|x_1 - y_1|$, $|y_1 - z_1|$ and $|z_1 - x_1|$, as x_2, y_2 and z_2 respectively, the absolute values of the differences of the numbers x_2, y_2 and z_2 as x_3, y_3 and z_3 respectively, and so on. It is known that for some n the triple of the numbers x_n, y_n and z_n coincides with the original triple of the numbers x, y and z. Find the numbers y and z.

132*. (a) There are four arbitrary positive integers A, B, C and D. Let us denote by A_1, B_1, C_1 and D_1 the differences between A and B, B and C, C and D and D and A (it is meant that every time we subtract a smaller number from a greater one). Then, proceeding from the numbers A_1, B_1, C_1 and D_1, we similarly form a 4-tuple of numbers A_2, B_2, C_2, and D_2, and so on. Prove that after the procedure has been repeated several times we must necessarily arrive at a 4-tuple of zeros.

For instance, starting with the numbers 32, 1, 110 and 7 we obtain, in succession,

32,	1,	110,	7
31,	109,	103,	25
78,	6,	78,	6
72,	72,	72,	72
0,	0,	0,	0

(b) Does the assertion stated in Problem (a) remain true when A, B, C and D are positive rational numbers and not necessarily integers? What is the answer to the same question when A, B, C and D are irrational numbers?

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*133**. **(a)** Arrange the numbers from 1 to 1000 as a sequence such that any 11 numbers (not necessarily consecutive members of the sequence) arbitrarily chosen from it do not form an increasing or a decreasing sequence.

(b) Prove that from any sequence formed by arranging in a certain way the numbers from 1 to 101 it is always possible to choose 11 numbers (which must not necessarily be consecutive members of the sequence) which form an increasing or a decreasing number sequence.

134. **(a)** Let there be 101 numbers arbitrarily chosen from the first 200 whole numbers 1, 2, ..., 200. Prove that among the chosen numbers there is a pair of numbers such that one of them is divisible by the other.

(b) Choose 100 numbers from the first 200 whole numbers so that none of them is divisible by any other.

(c) Prove that if at least one of 100 whole numbers not exceeding 200 is less than 16 then one of these numbers must necessarily be divisible by some other.

135. Prove that

(a) from any 52 integers it is always possible to choose two numbers such that their sum or difference is divisible by 100;

(b) from any 100 integers it is always possible to choose several numbers (or, perhaps, one number) whose sum is divisible by 100;

(c) if the numbers in Problem (b) are positive and do not exceed 100 and their sum is equal to 200 then it is possible to choose several numbers from them such that their sum is equal to 100;

(d)* from any 200 integers it is possible to choose 100 numbers whose sum is divisible by 100.

136. Let there be a nonincreasing sequence $a_1, a_2, \ldots, a_n$ of positive numbers whose sum is equal to 1, the greatest number in the sequence being equal to $\frac{1}{2k}$ where k is a whole number:

$$\frac{1}{2k}=a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \geqslant a_n > 0, \quad a_1+a_2+\ldots+a_n=1$$

Prove that it is possible to chose k numbers from this sequence such that the smallest of them exceeds half the greatest number.

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137. Let there be p crosses and q noughts written in a circular order. Let a denote the number of pairs of crosses standing side by side and b denote the number of pairs of noughts standing side by side. Prove that $a-b=p-q$.

138. Let $i_1, i_2, \ldots, i_n$ be a sequence of numbers $1, 2, \ldots, n$ which, in the general case, are arranged in some new order. Prove that for even n the product $(1-i_1)(2-i_2)(3-i_3)\ldots(n-i_n)$ can be even and can be odd and that for an odd n this product must necessarily be even.

139. Given n numbers $x_1, x_2, x_3, \ldots, x_n$ each of which is equal to $+1$ or to -1, prove that if $x_1x_2+x_2x_3+\ldots+x_{n-1}x_n+x_nx_1=0$ then n is divisible by 4.

140. Prove that the set of all whole numbers whose decimal representations involve only the digits 1 and 2 can be divided into two groups such that the decimal representation of the sum of any two numbers which belong to any of the groups involves not less than two digits 3.

141. There are five 100-digit numbers whose decimal representations involve only the digits 1 and 2. It is known that any two of the numbers have the same digits in exactly r of the 100 decimal places and that in no decimal place the corresponding five digits of the given five numbers coincide. Prove that this is only possible when r lies within the limits from 40 to 60: $40 \leqslant r \leqslant 60$.

142. There are two sets of the signs "+" and "—" each of which contains 1958 signs. It is allowed to perform repeatedly the operation of changing eleven signs arbitrarily chosen from the first set to the opposite. Prove that after a number of such operations it is possible to transform the first set into the second. (The sets are considered identical when they contain similar signs in the same places.)

143.* When training a chess-player plays at least one game of chess a day but in order to avoid overstrain he plays not more than 12 games a week. Prove that there must be a period of several consecutive days during which he plays exactly 20 games.

144. Let N be an arbitrary positive integer. Prove that there is a whole number multiple of N whose decimal representation is formed only of the digits 0 and 1. Besides, for the case when N is relative prime to 10 (that is when N is divisible neither by 2 nor by 5) prove that there exists a multiple of N whose decimal representation consists only of ones (if N is not relatively prime to 10 then, obviously, no number of the form $\underbrace{11\ldots1}_{n \text{ times}}$ can be divisible by N).

145. Construct a system of line segments lying on the number line and not overlapping one another (that is, having no common internal points and no common end points) each of which is of length 1 such that any (infinite) arithmetic progression (with an arbitrary first term and an arbitrary common difference!) contains at least one number falling inside one of the segments belonging to this system.

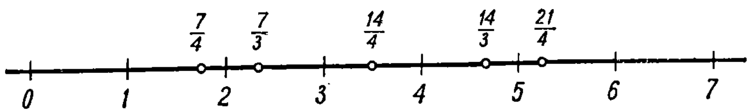

Fig. 2

146. Let m and n be two relatively prime positive integers. Prove that if the fractions

$$\frac{m+n}{m},\quad \frac{2(m+n)}{m},\quad \frac{3(m+n)}{m},\ \ldots,\ \frac{(m-1)(m+n)}{m}$$

and

$$\frac{m+n}{n},\quad \frac{2(m+n)}{n},\quad \frac{3(m+n)}{n},\ \ldots,\ \frac{(n-1)(m+n)}{n}$$

are represented by points on the number line then each of the intervals $(1, 2)$, $(2, 3)$, $(3, 4)$, $\ldots$, $(m+n-2, m+n-1)$ contains exactly one representing point (see Fig. 2 demonstrating the case when $m = 3$ and $n = 4$).

147*. Let $a_1, a_2, a_3, \ldots, a_n$ be arbitrary positive integers each of which is less than 1000. Let the least common multiple of any two of them be greater than 1000. Prove that the sum of the reciprocals of the numbers $a_1, a_2, a_3, \ldots, a_n$ is less than two.

148*. A fraction of the form q/p whose denominator is an odd prime number $p \neq 5$ is represented as an infinite repeating decimal. Prove that if the number of the digits in the period of the decimal is even then the arithmetic mean of all the digits forming the period is equal to 4.5 (this arithmetic mean thus coincides with the arithmetic mean of all the digits 0, 1, 2, ..., 9). This allows us to say that the "great" and the "small" digits are encountered in the period "equally frequently". Also prove that if the number of the digits forming the period is odd then the arithmetic mean of all these digits must necessarily be different from 4.5.

*149**. Let fractions of the form

$$\frac{a_1}{p}, \frac{a_2}{p^2}, \frac{a_3}{p^3}, \ldots, \frac{a_n}{p^n}, \ldots$$

(where p is a prime number different from 2 and 5 and $a_1, a_2, \ldots, a_n$ are arbitrary whole numbers relatively prime to p) be represented as infinite repeating decimals. Prove that the first several fractions (or, perhaps, one fracion) have the same number of digits in their periods and that for the other fractions the number of digits in the period of every decimal is p times as great as the number of digits in the period of the preceding decimal.

For instance, $\frac{1}{3}=0.\overline{3}$, $\frac{4}{9}=0.\overline{4}$, $\frac{10}{27}=0.\overline{370}$, $\frac{80}{81}=0.\overline{987654320}$, $\frac{116}{243}$ has 27 digits in the period, $\frac{653}{729}$ has 81 digits in the period and so on.

5. Finding Integral Solutions of Equations*

150. (a) Find a four-digit number which is a perfect square such that its first two digits are equal to each other and its last two digits are equal to each other.

(b) A sum of a two-digit number and a number represented with the aid of the same digits but written in the reverse order is a perfect square. Find all such numbers.

151. Find a 4-digit number which is equal to the square of the sum of two 2-digit numbers formed of the first two and the last two digits of the given number.

152. Find all 4-digit numbers which are perfect squares and whose decimal representations contain

(a) four even digits;
(b) four odd digits.

153. (a) Find all three-digit numbers equal to the sums of the factorials of their digits.

(b) Find all whole numbers equal to the sums of the squares of their digits.

* A method of finding integral solutions of certain algebraic equations is referred to as *Diophantine analysis* and the equations are termed *Diophantine equations* after Diophantus of Alexandria (c. 250 A. D.), a Greek algebraist. — *Tr.*

154. Find all whole numbers which are equal to

(a) the squares of the sums of their digits;

(b) the sums of the digits in the decimal representations of their cubes.

155. Find the integral solutions of the equations

(a) $1! + 2! + 3! + \ldots + x! = y^2$;

(b) $1! + 2! + 3! + \ldots + x! = y^z$.

156. In how many ways is it possible to represent 2^n as a sum of four squares of positive integers?

157. (a) Prove that the equality

$$x^2 + y^2 + z^2 = 2xyz$$

can hold for whole numbers x, y and z only when $x = y = z = 0$.

(b) Find the whole numbers x, y, z and v such that

$$x^2 + y^2 + z^2 + v^2 = 2xyzv$$

*158**. (a) For what integral values of k can the equality

$$x^2 + y^2 + z^2 = kxyz$$

hold where x, y and z are positive integers?

(b) Among the first thousand whole numbers find the possible triples of numbers for which the sums of their squares are divisible by their products.

159. Find the integral solutions of the equation

$$x^3 - 2y^3 - 4z^3 = 0$$

160. Find the integral solutions of the equation

$$x^2 + x = y^4 + y^3 + y^2 + y$$

161. Find the positive integral solutions of the equation

$$x^{2y} + (x+1)^{2y} = (x+2)^{2y}$$

162. Find the integral solutions of the equation

$$\underbrace{\sqrt{x + \sqrt{x + \ldots + x}}}_{y \text{ square roots}} = z$$

*163**. Prove that the equation $x^2 + x + 1 = py$ where the coefficient p is a prime number possesses integral solutions x, y for infinitely many values of p.

*164**. Find four positive integers such that the sum of the square of each of them and the remaining three numbers is a perfect square.

165. Find all the pairs of integers whose sums are equal to their products.

166. The sum of the reciprocals of three positive integers is equal to 1. Find these integers.

167. **(a)** Prove that the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$ where $n>1$ is a natural number has exactly three solutions x, y (where x and y are natural numbers) for any prime number n (solutions of the form $x=a$, $y=b$ and $x=b$, $y=a$ are considered to be different when $a\neq b$) and more than three such solutions for any composite number n.

(b) Find all integral solutions of Problem 167 (a) for $n=14$.

(c)* Find the integral solutions of the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ in x, y, z (derive the general formula expressing all the solutions).

168. **(a)** Find all pairs of positive integers x and y not equal to each other which satisfy the equation

$$x^y=y^x$$

(b) Find all pairs of positive rational numbers x and y not coinciding with each other which satisfy the equation

$$x^y=y^x$$

(derive the general formula expressing all such solutions).

169. Two pupils of the 5th form and several pupils of the 6th form participated in a chess tournament. Each pupil played once with every other participant. The two pupils of the 5th form together had 8 points; each of the pupils of the 6th form had one and the same number of points (in the tournament a winner receives 1 point, a loser receives 0 and for a drawn game each of the participants receives 1/2). How many pupils of the 6th form participated in the tournament?

170. Pupils of the 5th and of the 6th form took part in a chess tournament. Each participant played once with every other participant. The number of the pupils of the 6th form was 10 times that of the pupils of the 5th form and the number of points the former had together was 4.5 times that the pupils of the 5th form had. How many pupils of the 5th form participated in the tournament and how many points had they together?

171*. By an integer triangle we shall simply mean a triangle the lengths of whose sides are expressed by whole numbers. Find all integer triangles each of which has a perimeter equal to its area.

Problem 171 belongs to an important division of the theory of integral solutions of equations, and we do not go into detail here because this topic is very extensive. For instance, there are very interesting problems on integer triangles whose angles are commensurable with an angle of 360°. It can be proved that *an integer triangle can have no angles commensurable with an angle of* 360° *which are different from* 60°, 90° *and* 120°, and it is not difficult to derive formulas expressing the lengths of the sides of *all* integer triangles with a given angle α where α is equal to 60° or 90° or 120° (*right* integer triangles are often called *Pythagorean triangles*). It is also interesting to consider the problem of finding integer triangles whose two angles are in a given ratio, say one of them is twice or three times or five times or six times as great as the other. For instance, it can readily be proved that the *smallest integer triangle one of whose angles is twice as great as some other of its angles has sides of lengths* 4, 5 *and* 6 *and that the least possible lengths of the sides of an integer triangle one of whose angles is six times as great as some other of its angles are* 30 421; 46 656 and 72 930. Further, it is interesting to impose some definite conditions on the angles and on the sides of an integer triangle. For example, we can easily find *infinitely many Pythagorean triangles each of which has a hypotenuse or one of the legs expressed by perfect squares whereas there is no Pythagorean triangle the lengths of whose two sides are simultaneously perfect squares*. Besides, among the Pythagorean triangles each of which has a *hypotenuse expressed by a perfect square* there are infinitely many triangles *the sum of whose legs is also a perfect square*. The sides of all such triangles are very large: as early as 1643 P. Fermat * showed that *the smallest of the Pythagorean triangles satisfying the above conditions has sides whose lengths are*

$$a = 1\,061\,652\,293\,520$$

$$b = 4\,565\,486\,027\,761$$

and

$$c = 4\,687\,298\,610\,289$$

6. Matrices, Sequences and Functions

An $m \times n$ *matrix* (also called an *m-by-n matrix*) is simply a rectangular array of numbers having m horizontal rows and n columns. As examples, below are written a 2×4 matrix, a 3×3

* Pierre de Fermat (1602-1665), the great French mathematician, one of the founders of the number theory.

matrix (which is a *square matrix of the* 3*rd order*), a 3×1 matrix and a 1×5 matrix ($1 \times n$ matrices and $m \times 1$ matrices are also called *vectors*, the former being referred to as *row vectors* and the latter as *column vectors*):

$$\begin{bmatrix} 1/2 & -1/2 & 1/3 & 1/7 \\ 2/7 & -4/3 & 3/4 & 0 \end{bmatrix}; \quad \begin{bmatrix} 3 & -2 & 0 \\ -11 & 0 & 7 \\ 12 & 1 & -1 \end{bmatrix}; \quad \begin{bmatrix} \sqrt{2} \\ -\sqrt{3}/2 \\ 1 \end{bmatrix};$$

$$[10 \quad -9 \quad 8 \quad -7 \quad 6]$$

By an integer matrix we shall mean a matrix whose all elements are integers. For instance, such are the second and the fourth (but not the first and the third) of the matrices written above. We mention here the notion of a matrix because it plays an important role in mathematics.

A *number sequence* is a set of numbers ordered as are the positive integers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

(more often we deal with infinite sequences). To specify a sequence it is necessary to state a rule according to which its *members* (*elements*) $a_1, a_2, \ldots$ are formed. Such a rule can be expressed by a *formula* showing how an arbitrary element a_n can be computed for any given index n or by an *algorithm* which indicates some method with the aid of which a_n can be found for any concrete value of n. For instance, in several problems in the present section we shall encounter the *Fibonacci* * *sequence*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots \qquad (*)$$

(its members are the *Fibonacci numbers*). An algorithm determining this sequence ** is specified by the following rules:

$$a_1 = a_2 = 1; \quad a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n > 2 \qquad (**)$$

By the way, in mathematics we also encounter the so-called "random sequences" the formation of whose members is regulated by no strict rules (cf. Problem 192).

* Leonardo Fibonacci (Leonardo de Pisa) (1180-1240), a distinguished medieval European mathematician.

** There exist some other algorithms describing the Fibonacci numbers (*); for instance, rule (**) implies the formula

$$u_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

expressing the Fibonacci number u_n directly in terms of n.

By a *function* $y = f(x)$ is meant a law f: $x \mapsto y$ specifying a mapping of a set X of "admissible" values of the argument x onto a set Y of the values y of the function; to each value $x \in X$ there must correspond a single value $y = f(x)$.

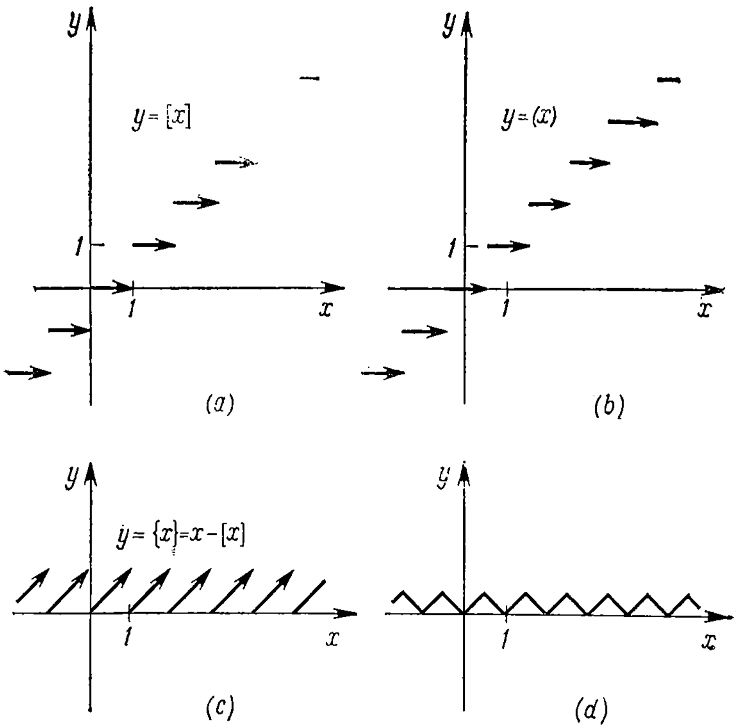

Fig. 3

In number theory those functions are most important which are connected with integers, that is those whose *domain* X and *range* Y consist of *integers*. When the domain $X = \{1, 2, 3, \ldots\}$ is the set of all natural numbers the argument x is more often denoted by the letter n; in this case a function $n \mapsto y(n)$ or $n \mapsto a_n$ simply reduces to a number sequence $\{a_1, a_2, a_3, \ldots\}$. A typical (and frequently encountered) example of a function whose range Y consists of integers is the so-called *integral part* $[x]$ of a number x which is defined as *the largest integer not exceeding* x (for instance, $[2.5] = 2$, $[4] = 4$ and $[-3.2] = -4$). Another function similar to $[x]$ is the one which we denote as (x): it is equal to *the nearest integer to* x, that is to the integer for which the absolute value of the difference between this integer and x assumes the smallest possible value (in case there are two such integers (x) is taken to be equal to the greatest of them; for instance, $(2.5)=3$, $(4)=4$ and $(-3.2)=-3$; compare the graphs of $[x]$ and of (x) shown in Fig. 3, *a* and *b*). In some mathematical problems we also deal with the function $\{x\}=x-[x]$ which is called the *fractional part* of the number x (see Fig. 3*c*). (For the sake of visuality, Fig. 3*d* shows the graph of the *deviation* of x from its *nearest integer*.) However, the most important role is played in the number theory by some "purely arithmetic" functions for which both the domains X and the ranges Y consist of integers. As examples we can mention some of such functions which are encountered in the problems below: *the number of the divisors* $\tau(n)=\tau_n$ of a (natural) number n, *the sum* $\sigma(n)=\sigma_n$ *of the divisors of* n *and the Möbius* * *function* $\mu(n)$ defined by the rule: $\mu(1)=1$, $\mu(n)=(-1)^k$ if $n=p_1p_2\ldots p_k$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct positive prime numbers and $\mu(n)=0$ for all the other positive integers n multiple of at least one square of a natural number; the Möbius function implicitly takes part in the solution of Problem 197. (It should be noted that all the three functions $\tau(n)$, $\sigma(n)$ and $\mu(n)$ possess the so-called "multiplication property": if $\varphi(n)$ is any of these functions then $\varphi(n_1n_2)=\varphi(n_1)\varphi(n_2)$ for any *relatively prime* natural numbers n_1 and n_2.)

The problems collected in this section are rather versatile both in their content and in the methods of their solution. In particular, in many of the problems the set of all points in the plane with integral coordinates is used; its application to number-theoretic problems was initiated in the works of H. Minkowski ** and G.F. Voronoi ***.

172. The numbers $1, 2, 3, \ldots, n^2$ are arranged as a square table of the form

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ n+1 & n+2 & n+3 & \ldots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \ldots & 3n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1)n+1 & (n-1)n+2 & (n-1)n+3 & \ldots & n^2 \end{bmatrix}$$

* Augustus Ferdinand Möbius (1790—1868), a distinguished German mathematician whose primary field of interest was geometry.

** Hermann Minkowski (1864—1906), a distinguished German mathematician who contributed much to geometry, physics (relativity theory) and number theory; he was one of the founders of the "geometrical theory of numbers".

*** G. F. Voronoi (1868-1908), a distinguished Russian mathematician, one of the founders of the "geometrical theory of numbers".

From this table a number is chosen and the row and the column containing this number are deleted. Then from the remaining number array one more number is chosen and again the row and the column containing this number are deleted and so forth until there remains only one number in the table which is automatically added to the set of the numbers chosen previously. Find the sum of all the numbers thus chosen.

173. A square table with n^2 cells is filled with integers assuming the values from 1 to n so that in each row and in each column there are all numbers from 1 to n. Prove that if the original table is symmetric about the diagonal joining its left upper corner and its right lower corner and if the number n is odd then there are all numbers from 1 to n on this diagonal. Does this assertion remain true for the case when n is an even number?

174. There are n^2 numbers from 1 to n^2 which are arranged to form a square table of dimension $n \times n$ so that the number 1 occupies an arbitrary place, the number 2 belongs to the row with serial number equal to that of the column containing the number 1, the number 3 belongs to the row with serial number coinciding with that of the column containing the number 2 and so on. What is the difference between the sum of the numbers belonging to the row containing the number 1 and the sum of the numbers belonging to the column containing the number n^2?

175. There is a rectangular table of dimension $m \times n$ in whose all cells some numbers are written. We are allowed to change to the opposite the signs of all numbers belonging to one row or of all numbers belonging to one column. Prove that on repeating these admissible operations several times we can always arrive at a table for which the sum of the numbers in each row and the sum of the numbers in each column are nonnegative.

176. 800 numbers are written to form a rectangular table of 100 rows and 80 columns so that the product of all numbers belonging to any column by the sum of all numbers belonging to any row is equal to the number standing at the intersection of this row and this column. It is known that the number standing in the right upper corner of the table is positive. Find the sum of all the numbers the table is formed of.

177. Sixty-four nonnegative numbers whose sum is equal to 1956 are written in the 64 squares of a chess-board. It is known that the sum of the numbers belonging to each of the two diagonals of the board is equal to 112 and that the numbers occupying any two squares symmetric about any of the diagonals are equal to each other. Prove that the sum of the numbers belonging to any row or to any column of the board is less than 518.

178. In the squares of a board (resembling a chess-board) of dimension $n \times n$ some numbers are written so that for any arrangement of n rooks on the board satisfying the condition that any two rooks cannot take each other the sum of the numbers standing in the squares occupied by the rooks is one and the same. Let a_{ij} denote the number placed at the intersection of the ith row of the board and the jth column. Prove that there exist two sets of numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ such that $a_{ij} = x_i + y_j$.

179. Some numbers are written in the squares of a chess-board of dimension $n \times n$. Let x_{pq} denote the number in the intersection of the pth row and the qth column. Prove that if for any i, j and k (where $1 \leqslant i, j, k \leqslant n$) there holds the identity $x_{ij} + x_{jk} + x_{ki} = 0$ then there exist n numbers $t_1, t_2, \ldots, t_n$ such that $x_{ij} = t_i - t_j$.

180. Stars are written in some of the squares of a chess-board of dimension $n \times n$. It is known that after an arbitrary number of rows of the board have been deleted (but, of course, not all the rows!) there remains a column containing exactly one star that has not been deleted. (In particular, if none of the rows is deleted then there is also a column containing exactly one star.) Prove that if an arbitrary number of columns has been deleted (but not all of them) then there remains a row containing exactly one star that has not been deleted.

181*. In all the squares of a chess-board of dimension $n \times n$ except one of them the signs "+" are written and the exceptional square contains the sign "—". Let us consider two cases when it is known that

(a) $n = 4$ and the sign "—" stands on a side of the board but not in its corner;

(b) $n = 8$ and the sign "—" is not placed in a corner of the board.

We are allowed to change simultaneously to the opposite all signs belonging to one (arbitrarily chosen) column or to one (also arbitrarily chosen) row or to one (arbitrarily chosen) "inclined line" parallel to one of the two diagonals of the board (in particular, as such an "inclined line" we can take one of the diagonals of the board or any corner square). Prove that we cannot get rid of the sign "—" by repeating these "admissible" changes of signs any number of times, that is we cannot arrive at the case when there are only the signs "+" in all the squares of the board.

182. **(a)** In all the squares of an ordinary chess-board of dimension 8×8 the signs "+" or "—" are placed. We are allowed to choose an arbitrary smaller quadratic array of squares of the board of dimension 3×3 or 4×4 with sides parallel to the sides of the board and to change to the opposite all the signs in the squares of such a rectangle. We can try to perform such operations repeatedly a number of times in order to arrive at an arrangement of signs on the board involving the signs "+" solely. Is it always possible?

(b) Some natural numbers are written in all the squares of an ordinary chess-board. We are allowed to increase by unity all numbers placed in the squares forming a smaller quadratic array consisting of four squares of the board adjoining one another or all numbers placed in (any) two neighbouring rows of the board or all numbers in (any) two neighbouring columns of the board. Is it always possible to perform a number of such operations in such a way that we arrive at the case when all the numbers on the board are divisible by 10?

183. **(a)** There are three sets of balls. We are allowed to take simultaneously one ball from each of the three sets or to duplicate the number of the balls in one (arbitrary) set. Is it possible to perform such operations several times so that all the balls are taken from all the three sets?

(b) In all the cells in a rectangular table of 8 rows and 5 columns some natural numbers are written. We are allowed to duplicate any number in any column or to subtract unity from all numbers of one (arbitrary) row. Prove that it is possible to perform a number of these "admissible" operations on the table in such a way that all the numbers in all the places of the table become equal to zero.

184. A table of positive integers having two columns and a number of rows is formed according to the following rule. In the upper row we write two arbitrary positive integers a and b, then under a we write a (positive) integer a_1 which is equal to $a/2$ if the number a is even and to $(a-1)/2$ if a is odd and under b we write $b_1 = 2b$. Next we perform on the numbers a_1 and b_1 the same operations as those performed on a and b, that is under a_1 we write a number a_2 equal to $a_1/2$ for an even a_1 and to $(a_1-1)/2$ for an odd a_1 and under b_1 we write $b_2 = 2b_1$. Further, under the numbers a_2 and b_2 we write new numbers a_3 and b_3 which are obtained from a_2 and b_2 in the way in which a_2 and b_2 were obtained from a_1 and b_1, etc. This process of repeated operations is stopped when we arrive at a number $a_n = 1$ (to which a number $b_n = 2b_{n-1}$ corresponds). Prove that the sum of all numbers b_i in the right column to which *odd* numbers a_i correspond is equal to the product ab (here i can assume any value from 0 to n; by a_0 and b_0 are meant the original numbers a and b respectively).

---

185. Prove that every natural number is either a Fibonacci number (that is a member of the Fibonacci sequence (*); see page 35) or can be represented in the form of a sum of several (distinct) Fibonacci numbers.

186. Prove that there are not eight consecutive Fibonacci numbers (see Problem 185) whose sum is not a Fibonacci number.

187. Prove that if a natural number n is divisible by 5 then the nth member u_n of the Fibonacci sequence (see Problem 185) is also divisible by 5.

188. Is there a number among the first 100 000 001 Fibonacci numbers (see Problem 185) whose decimal representation has four noughts at the end?

189. Let us consider a number sequence $a_1, a_2, a_3 \ldots$ constructed according to the following rule: $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{a_{n-1}}$ for $n > 1$. Prove that $14 < a_{100} < 18$.

190. A number sequence $a_1, a_2, a_3, \ldots, a_n$ is such that $a_1 = 0$, $|a_2| = |a_1 + 1|$, $|a_3| = |a_2 + 1|, \ldots, |a_n| = |a_{n-1} + 1|$. Prove that the arithmetic mean $(a_1 + a_2 + \ldots + a_n)/n$ of these numbers is not less than $-1/2$.

191.* A sequence of natural numbers $a_0, a_1, a_2, a_3, \ldots$ is formed according to the following rule:

$$a_0 a_1 a_2 = |a_0 - a_1|, \quad a_3 = |a_1 - a_2|, \ldots$$

(generally, $a_n = |a_{n-2} - a_{n-1}|$ for all $n \geqslant 2$). The elements of the sequence are computed until the first zero has been obtained. It is known that each of the numbers contained in the sequence does not exceed 1967. What is the greatest number of terms which such a sequence may contain?

192. Given an infinite sequence of digits $\alpha_1\alpha_2\alpha_3\alpha_4\ldots$ in which each of the digits can be equal to an arbitrary decimal digit except nine. Prove that among the numbers α_1; $\overline{\alpha_1\alpha_2}$; $\overline{\alpha_1\alpha_2\alpha_3}$; $\overline{\alpha_1\alpha_2\alpha_3\alpha_4}$; ... (here by $\overline{\alpha_1\alpha_2}$ is meant the number $\alpha_1 \cdot 10 + \alpha_2$ and the like) there are infinitely many composite numbers.

193. In a sequence 1975... each of the numbers, beginning with the fifth one, is equal to the last digit of the decimal representation of the sum of the four foregoing digits. Is it possible that

(a) the sequence contains a group of the four consecutive digits 1234?

(b) the four-tuple 1975 of the digits is again repeated in the sequence?

194. All the integer multiples of 9 are written as a sequence of the form

$$9;\ 18;\ 27;\ 36;\ 45;\ 54;\ 63;\ 72;\ 81;\ 90;\ 99;\ 108;\ 117;\ \ldots \qquad (*)$$

and for each of these numbers the sum of its digits is found:

$$9;\ 9;\ 9;\ 9;\ 9;\ 9;\ 9;\ 9;\ 9;\ 9;\ 18;\ 9;\ 9;\ \ldots \qquad (**)$$

In what place in sequence (**) does the number 81 first appear? What is the number following the first number 81? What occurs earlier in the sequence: the appearance of 4 consecutive numbers 27 or the appearance of 3 consecutive numbers 36? What else can you say about the alternation of the numbers in sequence (**)?

195. Let us consider the following sequence of collections of (natural) numbers. The initial collection I_0 consists of two unities: 1 and 1. Then, to obtain the following collection I_1, we insert between the numbers forming the initial collection their sum $1 + 1 = 2$, i.e. I_1 consists of the numbers 1, 2 and 1. Next we insert between every two numbers belonging to the collection I_1 their sum to obtain the collection I_2 consisting of the numbers 1, 3, 2, 3 and 1. Further, on performing the same operation on the collection I_2 we arrive at the collection I_3: 1, 4, 3, 5, 2, 5, 3, 4 and 1, etc. How many times is the number 1973 repeated in the millionth collection $I_{1\,000\,000}$?

196. There is a (finite) sequence of noughts and ones such that all the 5-tuples of consecutive digits which can be selected from the sequence are distinct (the 5-tuples can, of course, overlap; for instance, they can be like the 5-tuples 01011 and 01101 placed as ...0101101...). Prove that if the sequence cannot be continued with the preservation of the indicated property then the first four digits of the given sequence coincide with its last four digits.

197. All the divisors of the number $N = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37$ are written in one row. Under the divisor 1 and under those divisors which are products of an even number of prime factors the numbers $+1$ are written and under the divisors which are products of an odd number of prime factors the numbers -1 are written. Prove that the sum of all the numbers written in the lower row is equal to zero.

198. Let p and q be two relatively prime natural numbers. We shall call a natural number n "good" if it can be represented in the form $px + qy$ where x and y are nonnegative integers and "bad" if otherwise.

(a) Prove that there exists a number A such that if a sum of two integers is equal to A then one of them is necessarily "good" whereas the other is "bad".

(b) Given two relatively prime natural numbers p and q, it is required to determine the number of all the possible "bad" natural numbers corresponding to p and q.

199. Prove that if n is a nonnegative integer then it can be uniquely represented in the form $n = [(x + y)^2 + 3x + y]/2$ where x and y are nonnegative integers.

200. Let t be an arbitrary positive number and let $d(t)$ denote the number of irreducible fractions p/q whose numerators p and denominators q do not exceed t. Find the sum

$$S = d\left(\frac{100}{1}\right) + d\left(\frac{100}{2}\right) + d\left(\frac{100}{3}\right) + \ldots + d\left(\frac{100}{99}\right) + d\left(\frac{100}{100}\right)$$

201. Prove the following properties of the integral part of a number (see page 36):

(1) $[x+y] \geqslant [x]+[y]$;

(2) $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right]$ where n is a whole number;

(3) $\left[x+\frac{1}{2}\right]=[2x]-[x]$;

(4) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$.

202. Simplify the expression

$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots+\left[\frac{n+2^k}{2^{k+1}}\right]+\ldots$$

where n is a positive integer.

203*. Prove that if p and q are relatively prime whole numbers then

$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+\left[\frac{3p}{q}\right]+\ldots+\left[\frac{(q-1)p}{q}\right]=$$
$$=\left[\frac{q}{p}\right]+\left[\frac{2q}{p}\right]+\left[\frac{3q}{p}\right]+\ldots+\left[\frac{(p-1)q}{p}\right]=\frac{(p-1)(q-1)}{2}$$

204. Prove that

(a) $\tau_1+\tau_2+\tau_3+\ldots+\tau_n=\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\ldots+\left[\frac{n}{n}\right]$

where n is *a* natural number and τ_n is the number of the divisors of n.

(b) $\sigma_1+\sigma_2+\sigma_3+\ldots+\sigma_n=\left[\frac{n}{1}\right]+2\left[\frac{n}{2}\right]+3\left[\frac{n}{3}\right]+\ldots+n\left[\frac{n}{n}\right]$

where n is a natural number and σ_n is the sum of the divisors of n.

205. Is there a positive integer n such that the fractional part of the number $(2+\sqrt{2})^n$ (see page 37) exceeds 0.999999, that is

$$\{(2+\sqrt{2})^n\}=(2+\sqrt{2})^n-[(2+\sqrt{2})^n]>0.999999?$$

206*. (a) Prove that for any positive integer n the number $[(2+\sqrt{3})^n]$ is odd.

(b) Find the highest power of 2 by which the number $[(1+\sqrt{3})^n]$ is divisible.

207. Prove that if p is a prime number greater than 2 then the difference

$$[(2+\sqrt{5})^p]-2^{p+1}$$

is divisible by p.

208*. Prove that if p is a prime number then the difference

$$C(n, p) - \left[\frac{n}{p}\right]$$

is divisible by p where n is an arbitrary positive integer not less than p and $C(n, p)$ is the number of combinations of n things taken p at a time ($C(n, p)$ is also denoted as C_p^n or ${}_nC_p$ or $\binom{n}{p}$ and is called a *binomial coefficient*).

For instance, $C(11, 5) = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 462$, and the number $C(11, 5) - [11/5] = 462 - 2 = 460$ is divisible by 5.

209. Find all numbers α such that the numbers $[\alpha]$, $[2\alpha]$, $[3\alpha]$, ..., $[N\alpha]$ where N is a fixed natural number are all distinct and the numbers $[1/\alpha]$, $[2/\alpha]$, $[3/\alpha]$, ..., $[N/\alpha]$ are also all distinct.

In Problem 209 it is required to find a number α such that the numbers $[\alpha]$, $[2\alpha]$, $[3\alpha]$, ..., $[N\alpha]$ are distinct and the numbers $[\beta]$, $[2\beta]$, $[3\beta]$, ..., $[N\beta]$ are distinct where $\beta = 1/\alpha$. A more intricate problem of this kind is to find two numbers α and β (it is no longer required that $\beta = 1/\alpha$) such that the *infinite* sequences $[\alpha]$, $[2\alpha]$, $[3\alpha]$, ... and $[\beta]$, $[2\beta]$, $[3\beta]$, ... consist of pairwise distinct numbers. It can be proved that *these sequences contain all the natural numbers and that each natural number is involved exactly once in them if and only if α is an irrational number and* $1/\alpha + 1/\beta = 1$.

210*. Prove that in the equality

$$N = \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \ldots + \frac{N}{2^n} + \ldots$$

where N is an arbitrary positive integer it is possible to replace all the fractions by their nearest integers:

$$N = (N/2) + (N/4) + (N/8) + \ldots + (N/2^n) + \ldots$$

(on the terminology and notation see page 36).

7. Estimating Sums and Products

211. How many digits does the decimal representation of the number 2^{100} contain?

212. (a) Prove that

$$\frac{1}{15} < \frac{1}{10\sqrt{2}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100} < \frac{1}{10}$$

(b*) Prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{99}{100}<\frac{1}{12}$$

Remark. The result established in Problem 212 (b) obviously strengthens the result of Problem 212 (a).

213. Which of the two numbers 31^{11} and 17^{14} is greater?

214. Which of the two numbers below is greater?

(a) $A=2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$ and $B=3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}$ where the expression of A involves 1001 twos and the expression of B involves 1000 threes;

(b) B (see Problem 214 (a)) and $C=4^{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}$ where the expression of C involves 999 fours.

In this problem by an expression of the form $a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_{n-1}^{a_n}}}}}}}$ is always meant the number $a_1^{\left(a_2^{\left(a_3^{\cdot^{\cdot^{\cdot^{\left(a_{n-1}^{a_n}\right)}}}}\right)}\right)}$.

215. Prove that in the decimal number system the representations of the numbers 1974^n and 1974^n+2^n contain the same number of digits for any natural number n.

216. Among all the differences of the form 36^m-5^n where m and n are natural numbers find the one having the smallest absolute value.

217. Prove that

$$\frac{2^{100}}{10\sqrt{2}}<C(100,\,50)<\frac{2^{100}}{10}$$

where $C(100, 50)$ is the number of combinations of 100 things taken 50 at a time.

218. Which of the two numbers 99^n+100^n and 101^n is greater (here n is a positive integer)?

219. Which of the two numbers 100^{300} and $300!$ is greater?

220. Prove that for any positive integer n we have

$$2\leqslant\left(1+\frac{1}{n}\right)^n<3$$

221. Which of the two numbers $(1.000\,001)^{1\,000\,000}$ and 2 is greater?

222. Which of the two numbers 1000^{1000} and 1001^{999} is greater?

223. Prove that for any integer $n>6$ we have

$$\left(\frac{n}{2}\right)^n>n!>\left(\frac{n}{3}\right)^n$$

*224**. Prove that for any $m > n$ where m and n are positive integers we have

(a) $\left(1+\frac{1}{m}\right)^m > \left(1+\frac{1}{n}\right)^n$.

For instance, $(1+1/2)^2 = 9/4 = 2\frac{1}{4}$ and $(1+1/3)^3 = 64/27 = 2\frac{10}{27} > 2\frac{1}{4}$.

(b) $\left(1+\frac{1}{m}\right)^{m+1} < \left(1+\frac{1}{n}\right)^{n+1}$ for $n \geqslant 2$.

For example, $(1+1/2)^3 = 27/8 = 3\frac{3}{8}$ and $(1+1/3)^4 = 256/81 = 3\frac{13}{81} < 3\frac{3}{8}$.

Remark. As it follows from Problem 224 (a), every number in the sequence $(1+1/1)^1$, $(1+1/2)^2$, $(1+1/3)^3$, ..., $(1+1/n)^n$, ... is *greater than* the preceding one. On the other hand, none of these numbers exceeds 3 (see Problem 220), and therefore the expression $(1+1/n)^n$ tends to a definite limit as $n \to \infty$ (it is evident that this limit lies between 2 and 3). This limit is denoted as e; the approximate value of the number e accurate to 15 decimal places is 2.718281828459045.

Analogously, Problem 224 (b) implies that in the number sequence $(1+1/2)^3$, $(1+1/3)^4$, $(1+1/4)^5$, ..., $(1+1/n)^{n+1}$ every term is *less than* the foregoing one, and since all the terms are greater than 1 it follows that for n increasing indefinitely the expression $(1+1/n)^{n+1}$ tends to a definite limit. At the same time, the terms of the former sequence $(1+1/2)^2$, $(1+1/3)^3$, ..., $(1+1/n)^n$, ... tend to the corresponding terms of the latter sequence for $n \to 0$ because the difference between the ratio $(1+1/n)^{n+1}/(1+1/n)^n = 1+1/n$ and 1 is equal to $1/n$ and it decreases indefinitely as $n \to \infty$. Consequently, the limit of the latter sequence must be equal to the same number e. The number e plays an extremely important role in mathematics and is encountered in various problems (for instance, see Problem 225 or the remarks to Problems 231 and 234).

225. Prove that for any integer n exceeding six there hold the inequalities

$$\left(\frac{n}{e}\right)^n < n! < n\left(\frac{n}{e}\right)^n$$

where $e = 2.71828\ldots$ is the limit of the expression $(1+1/n)^n$ for $n \to \infty$.

This assertion strengthens the result established in Problem 223. In particular, it implies that, *given any two numbers* a_1 *and* a_2 *such that* $a_1 < e < a_2$ (for instance, $a_1 = 2.7$ and $a_2 = 2.8$ or $a_1 = 2.71$ and $a_2 = 2.72$ or $a_1 = 2.718$ and $a_2 = 2.719$ etc.), *we have the inequality*

$$\left(\frac{n}{a_1}\right)^n > n! > \left(\frac{n}{a_2}\right)^n$$

for all values of n *exceeding a definite number* (this number is different for different values of a_1).

Thus, the number e serves as the "boundary" which separates the numbers a such that $(n/a)^n$ increases "faster" than $n!$ for $n \to \infty$ from the numbers a such that $(n/a)^n$ increases "slower" than $n!$ (the existence of this boundary follows from Problem 223).

Indeed, $(n/a_2)^n < n!$ for any n exceeding 6 (because $a_2 > e$ and, according to Problem 225, $n! > (n/e)^n$ for $n > 6$). Further, from the results established in Problems 220 and 224 it follows that for $n \geqslant 3$ there hold the inequalities

$$n > e > \left(1 + \frac{1}{n}\right)^n = \frac{(n+1)^n}{n^n}, \quad n^{n+1} > (n+1)^n \text{ and } \sqrt[n]{n} > \sqrt[n+1]{n+1}$$

and hence for $n \geqslant 3$ the expression $\sqrt[n]{n}$ decreases as n grows. It can readily be seen that for sufficiently large values of n the expression $\sqrt[n]{n}$ becomes arbitrarily close to unity; for instance, this follows from the fact that $\log \sqrt[10k]{10^k} = k/10^k$ becomes arbitrarily small for sufficiently large k. Now let us choose N so that the inequality $\sqrt[N]{N} < \frac{e}{a_1}$ holds; from this inequality it follows that for $n > N$ we must have $\sqrt[n]{n} < \frac{e}{a_1}$. By the result established in Problem 225, the last inequality implies that $n! < \left(\frac{n/e}{\frac{n}{\sqrt{n}}}\right)^n < (n/a_1)^n$.

The inequality of Problem 225 can be made considerably more precise. Namely, it can be shown that for large values of n the number $n!$ is approximately equal to $C\sqrt{n}\,(n/e)^n$ where C is a constant number equal to $\sqrt{2\pi}$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

(more precisely, it is possible to prove that *the ratio*

$$n!\Big/\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right]$$

tends to unity as n increases indefinitely).

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226. Prove that

$$\frac{1}{k+1}n^{k+1} < 1^k + 2^k + 3^k + \ldots + n^k < \left(1 + \frac{1}{n}\right)^{k+1}\frac{1}{k+1}n^{k+1}$$

where n and k are arbitrary positive integers.

Remark. In particular, from the result established in Problem 226 it follows that

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + \ldots + n^k}{n^{k+1}} = \frac{1}{k+1}$$

227. Prove that for any integer $n > 1$ we have

(a) $$\frac{1}{2} < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \frac{3}{4};$$

(b) $$1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} < 2.$$

*228**. (a) Find the integral part of the number

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{1\,000\,000}}$$

(b) Compute the sum

$$\frac{1}{\sqrt{10\,000}} + \frac{1}{\sqrt{10\,001}} + \frac{1}{\sqrt{10\,002}} + \ldots + \frac{1}{\sqrt{1\,000\,000}}$$

to an accuracy of 1/50.

*229**. Find the integral part of the number

$$\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}} + \ldots + \frac{1}{\sqrt[3]{1\,000\,000}}$$

230. (a) Compute the sum

$$\frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \ldots + \frac{1}{1000^2}$$

to an accuracy of 0.006.

(b) Compute the sum

$$\frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \ldots + \frac{1}{1000!}$$

to an accuracy of 0.000 000 015.

231. Prove that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

becomes greater than any given number N when the value of n is sufficiently large.

Remark. The result established in Problem 231 can be made considerably more precise. Namely, it is possible to show that for large n the sum $1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n$ differs very slightly from $\log_e n$ (where $e = 2.718\ldots$; see the remark after Problem 224 *. The logarithm $\log_e a$ of a number a is usually denoted ln a *). More precisely, it can be proved that the difference

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}-\ln n$$

(here $\ln n = \log_e n$) does not exceed unity for any n.

232. Prove that if we delete from the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}$$

all the terms the decimal representation of whose denominators contains the digit 9 then, for any n, the sum of the remaining terms will be less than 80.

233. **(a)** Prove that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots+\frac{1}{n^2}<2$$

for any n.

(b) Prove that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\frac{1}{n^2}<1\frac{3}{4}$$

for any n.

It is evident that the result established in Problem 233 (b) is stronger than the one expressed by the inequality in Problem 233 (a). A still stronger result is established in Problem 332. Namely, as follows from Problem 332, the sum

$$1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2}$$

is less than $\pi^2/6 = 1.6449340668\ldots$ for any n (but, at the same time, for any N less than $\pi^2/6$, say for $N = 1.64$ or $N = 1.644934$, it is possible to indicate a number n such that the sum $1 + 1/4 + 1/9 + \ldots + 1/n^2$ is *greater than* N).

234*. Let us consider the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\ldots+\frac{1}{p}$$

where the denominators of the fractions are all prime numbers from 2 to some prime number p inclusive. Prove that this sum can be made to exceed any preassigned number N (to this end it is only necessary to choose a sufficiently large prime number p).

Remark. The result established in Problem 234 can be considerably strengthened. Namely, it can be shown that for large p the difference between the sum $1 + 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + \ldots + 1/p$ and $\ln \ln p$ is comparatively small (as was mentioned, ln means the logarithm to base $e = 2.718\ldots$). More

* $\ln a = \log_e a$ is called the *natural logarithm of a* or the *Napierian logarithm of a* after J. Napier (1550-1617), the Scottish inventor of such logarithms. — *Tr.*

precisely, it is possible to prove that *the difference*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots+\frac{1}{p}-\ln\ln p$$

($\ln\ln p = \log_e\log_e p$) *does not exceed the number* 15.

It should also be noted that the comparison of the result established in Problem 234 with the results of Problems 232 and 233 allows us to say that there are "rather many" prime numbers in the sequence of all natural numbers (in particular, as follows from Problem 234, there is an *infinitude* of prime numbers). We can say that the prime numbers are encountered in the sequence of natural numbers "more frequently" than perfect squares or than those numbers whose decimal representations do not contain the digit 9 because, for instance, the sum of the reciprocals of all the squares of all natural numbers and the sum of the reciprocals of all natural numbers whose decimal representations do not involve the digit nine are bounded whereas the sum of the reciprocals of prime numbers can be made arbitrarily large.

8. Miscellaneous Problems in Algebra

Most of the problems collected in this book deal with arithmetical questions (and with some ideas of "higher arithmetic", that is number theory) but the problems in Secs. 8-10 are related to algebra and trigonometry. The solutions of some of these problems involve a number of rather important general notions. For instance, such are the so-called *fundamental theorem of algebra* asserting that *every polynomial* (algebraic) *equation of degree n* (with arbitrary real or complex coefficients) *has exactly n roots* (these roots can be real or complex numbers and must not necessarily be all pairwise distinct), *Vieta's* * *formulas* expressing the coefficients of an arbitrary algebraic equation in terms of its roots, the rule for long division of polynomials and geometrical representation of complex numbers.

235. Prove that

$$(a+b+c)^{333}-a^{333}-b^{333}-c^{333}$$

is divisible by

$$(a+b+c)^3-a^3-b^3-c^3$$

236. Factor the expression

$$a^{10}+a^5+1$$

237. Prove that the polynomial

$$x^{9999}+x^{8888}+x^{7777}+x^{6666}+x^{5555}+x^{4444}+x^{3333}+x^{2222}+x^{1111}+1$$

is divisible by

$$x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+1$$

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* Francois Vieta (1540-1603), a distinguished French mathematician, one of the creators of algebra and of modern algebraic notation.

238. (a) Factor the expression

$$a^3+b^3+c^3-3abc$$

(b) Using the result established in Problem 238 (a) derive the general formula for the solutions of the cubic equation

$$x^3+px+q=0$$

Remark. It should be noted that, proceeding from the result established in Problem 238, we can solve *any* equation of the third degree. Indeed, let

$$x^3+Ax^2+Bx+C=0$$

be an arbitrary cubic equation (given any algebraic equation of the third degree with an arbitrary nonzero coefficient in x^3, we can always bring it to the form in which the coefficient in x^3 is equal to 1; to this end we simply divide the whole equation by that leading coefficient). Let us make the substitution $x = y+c$ in the given equation. This yields

$$y^3+3cy^2+3c^2y+c^3+A(y^2+2cy+c^2)+B(y+c)+C=0$$

whence

$$y^3+(3c+A)y^2+(3c^2+2Ac+B)y+(c^3+Ac^2+Bc+C)=0$$

Now we put $c=-\frac{A}{3}$ (that is $x=y-A/3$) and thus arrive at an equation of the form

$$y^3+\left(\frac{3A^2}{9}-\frac{2A^2}{3}+B\right)y+\left(-\frac{A^3}{27}+\frac{A^3}{9}-\frac{AB}{3}+C\right)=0$$

which belongs to the type of the equations considered in Problem 238:

$$y^3+py+q=0$$

where $p=-\frac{A^2}{3}+B$ and $q=\frac{2A^3}{27}-\frac{AB}{3}+C$.

239. Solve the equation

$$\sqrt{a-\sqrt{a+x}}=x$$

240.* Find the real roots of the equation

$$x^2+2ax+\frac{1}{16}=-a+\sqrt{a^2+x-\frac{1}{16}}$$

where $0<a<1/4$.

241. Find the real roots of the equation

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{3x}}}}}}_{n \text{ radical signs}}=x$$

(all the square roots involved in the equation are meant to be positive).

242. Solve the equation

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{\ddots}{1+\cfrac{1}{x}}}}}=x$$

where in the expression on the left-hand side the sign of a fraction is repeated n times.

243. Find the real roots of the equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$$

(all the square roots are assumed to be positive).

244. Solve the equation

$$|x+1|-|x|+3|x-1|-2|x-2|=x+2$$

245. Solve the equation

$$1-\frac{x}{1}+\frac{x(x-1)}{1\cdot 2}-\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}+\ldots$$
$$\ldots+(-1)^n\frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}=0$$

246. Solve the equation $x^3-[x]=3$ where, as usual, $[x]$ denotes the *integral part* of the number x (see page 36).

247. In the general case the system of equations

$$\left.\begin{array}{r} x^2-y^2=0 \\ (x-a)^2+y^2=1 \end{array}\right\}$$

possesses four solutions. For what values of a does the number of the solutions of this system reduce to three or to two?

248. **(a)** Solve the system of equations

$$\left.\begin{array}{l} ax+y=a^2 \\ x+ay=1 \end{array}\right\}$$

For what values of a does the system have no solutions at all? For what values of a does it have infinitely many solutions?

(b) Solve the same problem for the system

$$\left.\begin{array}{l} ax+y=a^3 \\ x+ay=1 \end{array}\right\}$$

(c) Solve the same problem for the system

$$\left.\begin{aligned} ax+y+z&=1\\ x+ay+z&=a\\ x+y+az&=a^2 \end{aligned}\right\}$$

249. Find the conditions which should be satisfied by numbers α_1, α_2, α_3 and α_4 so that the system of six equations with four unknowns of the form

$$\left.\begin{aligned} x_1+x_2&=\alpha_1\alpha_2\\ x_1+x_3&=\alpha_1\alpha_3\\ x_1+x_4&=\alpha_1\alpha_4\\ x_2+x_3&=\alpha_2\alpha_3\\ x_2+x_4&=\alpha_2\alpha_4\\ x_3+x_4&=\alpha_3\alpha_4 \end{aligned}\right\}$$

is solvable. Find the values of the unknowns x_1, x_2, x_3 and x_4 for the case when these conditions (imposed on the numbers α_1, α_2, α_3 and α_4) hold.

250. Determine the number of real solutions of the system of equations

$$\left.\begin{aligned} x+y&=2\\ xy-z^2&=1 \end{aligned}\right\}$$

251. Find all real solutions of the system

$$\left.\begin{aligned} x^3+y^3&=1\\ x^4+y^4&=1 \end{aligned}\right\}$$

252. Find all the possible solutions x, x_1, x_2, x_3, x_4, x_5 of the simultaneous equations

$$x_1+x_3=xx_2,\quad x_2+x_4=xx_3,\quad x_3+x_5=xx_4,$$
$$x_4+x_1=xx_5,\quad x_5+x_2=xx_1$$

253. A 4-tuple of real numbers is such that the sum formed by each of the four numbers and the product of the other three numbers is equal to 2. Find all such 4-tuples.

254. Solve the following system of four equations with four unknowns:

$$\left.\begin{aligned} |a-b|y+|a-c|z+|a-d|t&=1\\ |b-a|x+|b-c|z+|b-d|t&=1\\ |c-a|x+|c-b|y+|c-d|t&=1\\ |d-a|x+|d-b|y+|d-c|z&=1 \end{aligned}\right\}$$

Here a, b, c, and d are some arbitrary pairwise distinct real numbers.

255. Consider the following system of n equations with the n unknowns $x_1, x_2, \ldots, x_n$:

$$ax_1^2 + bx_1 + c = x_2, \quad ax_2^2 + bx_2 + c = x_3, \quad \ldots$$
$$\ldots, \quad ax_{n-1}^2 + bx_{n-1} + c = x_n, \quad ax_n^2 + bx_n + c = x$$

where $a \neq 0$. Prove that this system possesses no solutions when $(b-1)^2 - 4ac < 0$, has a single solution for $(b-1)^2 - 4ac = 0$ and has more than one solution for $(b-1)^2 - 4ac > 0$.

256. Let $a_1, a_2 \ldots, a_n$ (where $n \geqslant 2$) be positive numbers. Determine the number of real solutions of the system of equations

$$x_1x_2 = a_1, \quad x_2x_3 = a_2, \quad \ldots, \quad x_{n-1}x_n = a_{n-1}, \quad x_nx_1 = a_n$$

257. **(a)** Determine the number of roots of the equation

$$\sin x = \frac{x}{100}$$

(b) Determine the number of roots of the equation

$$\sin x = \log x$$

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258. It is known that

$$\begin{aligned} a_1 - 4a_2 + 3a_3 &\geqslant 0, \\ a_2 - 4a_3 + 3a_4 &\geqslant 0, \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ a_{98} - 4a_{99} + 3a_{100} &\geqslant 0, \\ a_{99} - 4a_{100} + 3a_1 &\geqslant 0, \\ a_{100} - 4a_1 + 3a_2 &\geqslant 0. \end{aligned}$$

Let $a_1 = 1$; find the numbers $a_2, a_3, \ldots, a_{100}$.

259. Let a, b, c and d be four arbitrary positive numbers. Prove that the three inequalities

$$a + b < c + d$$
$$(a+b)(c+d) < ab + cd$$
$$(a+b)\,cd < (c+d)\,ab$$

cannot hold simultaneously.

260. Prove that the fraction

$$\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}$$

involving n radical signs in the numerator and $n-1$ radical signs in the denominator is greater than 1/4 for any $n \geqslant 1$.

261. The product of three given positive numbers is equal to 1; the sum of these numbers exceeds the sum of their reciprocals. Prove that one of the three numbers is greater than unity while the other two numbers are less than unity.

262. The sum of 1959 given positive numbers $a_1, a_2, a_3, \ldots, a_{1959}$ is equal to 1. Prove that the sum of all the possible products of 1000 different factors chosen from the set of these numbers is less than 1. (The set of the products under consideration includes all products which differ from one another *in at least one factor*; the products differing from one another only in the order in which the factors are multiplied are identified and only one of them is included in the sum in question.)

263. Let $N \geqslant 2$ be a natural number. Find the sum of all fractions of the form $1/mn$ where m and n are relatively prime natural numbers such that $1 \leqslant m < n \leqslant N$ and $m+n > N$.

264. Let 1973 positive numbers $a_1, a_2, a_3, \ldots, a_{1973}$ satisfy the condition

$$a_1^{a_2} = a_2^{a_3} = a_3^{a_4} = \ldots = (a_{1972})^{a_{1973}} = (a_{1973})^{a_1}$$

Prove that $a_1 = a_{1973}$.

265*. Prove that if x_1 and x_2 are the roots of the equation $x^2 - 6x + 1 = 0$ then, for any integral n, the number $x_1^n + x_2^n$ is an integer divisible by 5.

266. Let us consider the expression

$$(a_1 + a_2 + \ldots + a_{999} + a_{1000})^2 =$$
$$= a_1^2 + a_2^2 + \ldots + a_{999}^2 + a_{1000}^2 + 2a_1a_2 + 2a_1a_3 + \ldots + 2a_{999}a_{1000}.$$

where some of the numbers $a_1, a_2, \ldots, a_{999}, a_{1000}$ are positive while the others are negative. Is it possible that the number of positive pairwise products of different numbers in this expression is equal to the number of negative pairwise products?

Answer the same question for the expression

$$(a_1 + a_2 + \ldots + a_{9999} + a_{10\,000})^2$$

267. Prove that any integral power of the number $\sqrt{2}-1$ can be represented in the form $\sqrt{N}-\sqrt{N-1}$ where N is a whole number (for instance, $(\sqrt{2}-1)^2 = 3 - 2\sqrt{2} = \sqrt{9} - \sqrt{8}$ and $(\sqrt{2}-1)^3 = 5\sqrt{2} - 7 = \sqrt{50} - \sqrt{49}$).

268. Prove that the number $99\,999 + 111\,111\sqrt{3}$ cannot be represented in the form $(A + B\sqrt{3})^2$ where A and B are whole numbers.

269. Prove that $\sqrt[3]{2}$ cannot be written as $\sqrt[3]{2} = p + q\sqrt{r}$ where p, q, and r are rational numbers.

270. It is known that a number A can be written in the form $A = \left(\frac{n + \sqrt{n^2 - 4}}{2}\right)^m$ where m and $n \geqslant 2$ are natural numbers. Prove that A can also be represented as $A = \frac{k + \sqrt{k^2 - 4}}{2}$ where k is a natural number.

271. Are there rational numbers x, y, z and t such that

$$(x + y\sqrt{2})^{2n} + (z + t\sqrt{2})^{2n} = 5 + 4\sqrt{2}$$

for some natural number n?

272. Suppose that there are two barrels of infinite volumes filled with water. Is it possible to pour exactly one litre of water from one of them into the other using two scoops of volumes $\sqrt{2}$ and $2 - \sqrt{2}$ litres respectively?

273. For what rational values of x is the expression $3x^2 - 5x + 9$ equal to the square of a rational number?

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274. The magnitude of the discriminant $\Delta = p^2 - 4q$ of a quadratic equation $x^2 + px + q = 0$ is of the order of 10. Prove that when the coefficient q of the equation is rounded so that its variation is of the order of 0.01 the increments of the values of the roots of the equation are of the order of 0.001.

275. Let us agree to *round* numbers by replacing them by integers differing from the original numbers by less than 1. Prove that any n positive numbers can be rounded in this way so that the sum of any of these numbers differs from the sum of the corresponding rounded numbers by not more than $(n + 1)/4$.

276. Let a be a positive number. This number is replaced by a number a_0 obtained by discarding all the digits in the decimal representation of a beginning with the fourth digit, that is the number a is rounded to its minor decimal approximation with an accuracy of 0.001. The number a_0 thus obtained is then divided by the number a itself and the quotient is again rounded in the same way to the same accuracy. Find all the numbers that can be obtained in this manner.

*277**. Let α be an arbitrary nonnegative irrational number and $n > 0$ be an arbitrary integer. Then in the sequence $0/n$, $1/n$, $2/n$, $3/n$, ... there is a fraction which is the closest to α, the absolute value of the difference between α and that fraction obviously being not greater than half the fraction $1/n$. Prove that there exists n such that the fraction with the denominator n which is the closest to α differs from α by less than $0.001 \cdot (1/n)$.

278. **(a)** Prove that if a number α has a decimal representation of the form 0.999... where there are 100 consecutive 9's after the decimal point then the decimal representation of $\sqrt{\alpha}$ has the form $\sqrt{\alpha} = 0.999\ \ldots$ where there are also 100 consecutive 9's after the decimal point.

(b*) Find the value of the root $\sqrt{0.\underbrace{1111\ \ldots\ 111}_{100 \text{ ones}}}$ to an accuracy of (1) 100; (2) 101; (3) 200 and (4) 300 decimal places after the decimal point.

279. **(a)** Which of the two numbers

$$\frac{2.00000000004}{(1.00000000004)^2 + 2.00000000004}$$

and

$$\frac{2.00000000002}{(1.00000000002)^2 + 2.00000000002}$$

is greater?

(b) Let $a > b > 0$. Which of the two numbers

$$\frac{1 + a + a^2 + \ldots + a^{n-1}}{1 + a + a^2 + \ldots + a^n} \quad \text{and} \quad \frac{1 + b + b^2 + \ldots + b^{n-1}}{1 + b + b^2 + \ldots + b^n}$$

is greater?

280. Given n numbers $a_1, a_2, a_3, \ldots, a_n$, find the number x such that the sum

$$(x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$$

assumes the least possible value.

281. **(a)** Given four real pairwise distinct numbers $a_1 < a_2 < a_3 < a_4$, it is required to arrange them in a certain order as $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ (where i_1, i_2, i_3 and i_4 are the same indices 1, 2, 3 and 4 but possibly rearranged in some way) so that the sum

$$\Phi = (a_{i_1} - a_{i_2})^2 + (a_{i_2} - a_{i_3})^2 + (a_{i_3} - a_{i_4})^2 + (a_{i_4} - a_{i_1})^2$$

takes on the minimum possible value.

(b*) Given n pairwise distinct numbers $a_1, a_2, a_3, \ldots, a_n$, it is required to arrange them in a certain order as $a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_n}$ to make the sum

$$\Phi = (a_{i_1} - a_{i_2})^2 + (a_{i_2} - a_{i_3})^2 + \ldots + (a_{i_{n-1}} - a_{i_n})^2 + (a_{i_n} - a_{i_1})^2$$

assume the least possible value.

282. **(a)** Prove that for arbitrary real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we always have

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \ldots + \sqrt{a_n^2 + b_n^2} \geqslant$$

$$\geqslant \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + (b_1 + b_2 + \ldots + b_n)^2}$$

Under what conditions does an exact equality take place here?

(b) A pyramid is said to be *right* if it is possible to inscribe a circle in the base of the pyramid and if the centre of the circle coincides with the foot of the altitude of the pyramid. Prove that a right pyramid has a smaller lateral area than any other pyramid of the same altitude the area and the perimeter of whose base coincide with those of the base of the former pyramid.

Remark. The inequality established in Problem 282 (a) is a special case of *Minkowski's inequality* which is written as

$$\sqrt{a_1^2+b_1^2+\ldots+l_1^2}+\sqrt{a_2^2+b_2^2+\ldots+l_2^2}+\ldots$$
$$\ldots+\sqrt{a_n^2+b_n^2+\ldots+l_n^2}\geqslant$$
$$\geqslant\sqrt{(a_1+\ldots+a_n)^2+(b_1+\ldots+b_n)^2+\ldots+(l_1+\ldots+l_n)^2}$$

283*. Prove that for any real numbers $a_1, a_2, \ldots, a_n$ there holds the inequality

$$\sqrt{a_1^2+(1-a_2)^2}+\sqrt{a_2^2+(1-a_3)^2}+\ldots$$
$$\ldots+\sqrt{a_{n-1}^2+(1-a_n)^2}+\sqrt{a_n^2+(1-a_1)^2}\geqslant\frac{n\sqrt{2}}{2}$$

For what values of these numbers is the left-hand member of the inequality exactly equal to its right-hand member?

284. Prove that if the absolute values of two numbers x_1 and x_2 do not exceed unity then

$$\sqrt{1-x_1^2}+\sqrt{1-x_2^2}\leqslant 2\sqrt{1-\left(\frac{x_1+x_2}{2}\right)^2}$$

For what values of x_1 and x_2 are the right-hand and the left-hand members of this inequality exactly equal to each other?

---

285. Which of the two expressions $\cos\sin x$ and $\sin\cos x$ has a greater value?

286. Prove without using a table of logarithms that

$$\text{(a)}\ \frac{1}{\log_2\pi}+\frac{1}{\log_5\pi}>2;$$

$$\text{(b)}\ \frac{1}{\log_2\pi}+\frac{1}{\log_\pi 2}>2.$$

287. Prove that if α and β are acute angles and $\alpha<\beta$ then

(a) $\alpha-\sin\alpha<\beta-\sin\beta$; **(b)** $\tan\alpha-\alpha<\tan\beta-\beta$.

288*. Prove that if α and β are acute angles and $\alpha<\beta$ then $\tan\alpha/\alpha<\tan\beta/\beta$.

289. Find the relation between $\arcsin\cos\arcsin x$ and $\arccos\sin\arccos x$.

290. Prove that it is impossible for the sum

$$\cos 32x + a_{31}\cos 31x + a_{30}\cos 30x + \ldots + a_2\cos 2x + a_1\cos x$$

to take on only positive values for all x whatever the coefficients $a_{31}, a_{30}, \ldots, a_2, a_1$.

291. Let some (or all) of numbers $a_1, a_2, \ldots, a_n$ be equal to $+1$ and the rest of them be equal to -1. Prove that

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{4} + \ldots + \frac{a_1a_2\ldots a_n}{2^{n-1}}\right)45^\circ =$$

$$= a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2 + \ldots + a_n\sqrt{2}}}}$$

For instance, for $a_1 = a_2 = \ldots = a_n = 1$ we obtain

$$2\sin\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{2}{2^{n-1}}\right)45^\circ = 2\cos\frac{45^\circ}{2^{n-1}} =$$

$$= \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n \text{ radical signs}}$$

9. Algebra of Polynomials

292. Find the sum of the coefficients of the polynomial obtained after parentheses have been removed and like terms have been collected in the product

$$(1 - 3x + 3x^2)^{743}(1 + 3x - 3x^2)^{744}$$

293. In which of the two polynomials obtained after parentheses have been removed and like terms have been collected in the expressions

$$(1 + x^2 - x^3)^{1000} \quad \text{and} \quad (1 - x^2 + x^3)^{1000}$$

is the coefficient in x^{20} greater?

294. Prove that the polynomial obtained after parentheses have been removed and like terms have been collected in the product

$$(1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}) \cdot (1 + x + x^2 + \ldots + x^{99} + x^{100})$$

does not involve terms with odd powers of x.

295. Find the coefficients in x^{50} in the polynomials obtained after parentheses have been removed and like terms have been collected in the expressions

(a) $(1 + x)^{1000} + x(1 + x)^{999} + x^2(1 + x)^{998} + \ldots + x^{1000}$;

(b) $(1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \ldots + 1000(1 + x)^{1000}$.

*296**. Determine the coefficient in x^2 appearing after parentheses have been removed and like terms have been collected in the expression

$$\underbrace{(\ldots(((x-2)^2-2)^2-2)^2-\ \ldots\ -2)^2}_{k\text{ times}}$$

297. Find the remainder obtained when the polynomial

$$x+x^3+x^9+x^{27}+x^{81}+x^{243}$$

is divided
(a) by $x-1$; (b) by x^2-1.

298. When an unknown polynomial is divided by $x-1$ and by $x-2$ we obtain in the remainder 2 and 1 respectively. Find the remainder resulting from the division of this polynomial by $(x-1)(x-2)$.

299. When the polynomial $x^{1951}-1$ is divided by $x^4+x^3+2x^2+x+1$ we obtain a quotient and a remainder. Find the coefficient in x^{14} in the quotient.

300. Find all polynomials $P(x)$ for which the identity

$$xP(x-1)\equiv(x-26)P(x)$$

holds.

301. Let us consider a polynomial $P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ whose coefficients $a_0, a_1, \ldots, a_n$ are
(a) some natural numbers; (b) arbitrary integers.

Let us denote by $s(n)$ the sum of the digits in the decimal representation of the number $P(n)$ (it is clear that the sum $s(n)$ only makes sense when $P(n)$ is a natural number; if otherwise, $a(n)$ simply does not exist).

Prove that if the sequence $s(1), s(2), s(3), \ldots$ contains infinitely many different numbers then it also contains infinitely many equal numbers.

302. Prove that the polynomial $x^{200}y^{200}+1$ in variables x and y cannot be written as a product $f(x)g(y)$ of two polynomials $f(x)$ and $g(y)$ depending solely on x and solely on y respectively.

303. A quadratic trinomial $p(x)=ax^2+bx+c$ is such that the equation $p(x)=x$ has no real roots. Prove that in this case the equation $p(p(x))=x$ has no real roots either.

304. A quadratic trinomial $p(x)=ax^2+bx+c$ is such that $|p(x)|\leqslant 1$ for $|x|\leqslant 1$. Prove that in this case from the condition $|x|\leqslant 1$ it also follows that $|p_1(x)|\leqslant 2$ where $p_1(x)=cx^2+bx+a$.

305. Prove that if x_1 is a root of an equation of the form

$$ax^2+bx+c=0 \qquad (1)$$

and x_2 is a root of the equation

$$-ax^2+bx+c=0 \qquad (2)$$

then there is a root x_3 of the equation

$$\frac{a}{2}x^2 + bx + c = 0 \tag{3}$$

lying between x_1 and x_2, that is $x_1 \leqslant x_3 \leqslant x_2$ or $x_1 \geqslant x_3 \geqslant x_2$.

306. Let α and β be the roots of an equation

$$x^2 + px + q = 0$$

and let γ and δ be the roots of an equation

$$x^2 + Px + Q = 0$$

Express the product

$$(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$$

in terms of the coefficients of the given equations.

307. For the two equations

$$x^2 + ax + 1 = 0 \quad \text{and} \quad x^2 + x + a = 0$$

determine all the values of the coefficient a for which these equations have at least one common root.

308. **(a)** Find an integer a such that

$$(x - a)(x - 10) + 1$$

can be factored as a product $(x + b)(x + c)$ of two factors involving integral numbers b and c.

(b) Find all nonzero and pairwise different integers a, b and c such that the polynomial

$$x(x - a)(x - b)(x - c) + 1$$

of the fourth degree with integral coefficients can be represented as a product of two polynomials with integral coefficients.

309. For what pairwise distinct integral coefficients $a_1, a_2, \ldots, a_n$ can the polynomials

(a) $(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n) - 1$

and

(b) $(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n) + 1$

be represented as products of some other polynomials?

310*. Prove that for any pairwise distinct integers $a_1, a_2, \ldots, a_n$ the polynomial

$$(x - a_1)^2(x - a_2)^2 \ldots (x - a_n)^2 + 1$$

cannot be represented as a product of two other polynomials with integral coefficients.

311. Prove that if a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

with integral coefficients assumes the value 7 for four integral values of x then it cannot take the value 14 for any integral value of x.

312. Prove that if a polynomial of the 7th degree

$$a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7$$

takes on the values $+1$ and -1 for 7 integral values of x then it cannot be represented as a product of two polynomials with integral coefficients.

313. Prove that if a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

with integral coefficients assumes odd values for $x = 0$ and $x = 1$ then the equation $P(x) = 0$ possesses no integral roots.

314*. Prove that if the absolute value of a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

with integral coefficients is equal to 1 for two integral values $x = p$ and $x = q$ $(p > q)$ of the argument and if the equation $P(x) = 0$ has a rational root a, then $p - q$ is equal to 1 or 2 and $a = (p + q)/2$.

315*. Prove that the polynomials

(a) $$x^{2222} + 2x^{2220} + 4x^{2218} + 6x^{2216} + 8x^{2214} + \dots$$
$$\dots + 2218x^4 + 2220x^2 + 2222;$$

and

(b) $$x^{250} + x^{249} + x^{248} + x^{247} + x^{246} + \dots + x^2 + x + 1$$

cannot be represented as products of polynomials with integral coefficients.

316. Prove that if a product of two polynomials with integral coefficients is equal to a polynomial with even coefficients which are not all divisible by 4 then all coefficients of one of the original polynomials are even whereas not all coefficients of the other polynomial are even.

317. Prove that all rational roots of a polynomial

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

with integral coefficients (the coefficient in the highest power of x is equal to 1) are integers.

318*. Prove that there exists no polynomial of the form

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

for which all the values $P(0)$, $P(1)$, $P(2)$ ... are prime numbers.

Remark. L. Euler * was the first to prove the assertion stated in this problem. He also constructed examples of polynomials whose values corresponding to many consecutive integers x are prime numbers (for instance, in the case of the polynomial $P(x) = x^2 - 79x + 1601$ the 80 values $P(0) = 1601$, $P(1) = 1523$, $P(2)$, $P(3)$, ..., $P(79)$ are prime numbers).

319. Prove that if a polynomial

$$P(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \dots + A_{n-1}x + A_n$$

possesses the property that it assumes integral values for all integral values of x, then it can be represented in the form of a sum of the polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{x(x-1)}{1 \cdot 2}, \dots$$

$$\dots, \quad P_n(x) = \frac{x(x-1)(x-2)\dots(x-n+1)}{1 \cdot 2 \cdot 3 \dots n}$$

multiplied by some integral factors. (According to Problem 75 (a), each of the polynomials $P_0(x)$, ..., $P_n(x)$ possesses the same property.)

320. **(a)** Prove that if a polynomial $P(x)$ of the nth degree assumes integral values for $x = 0, 1, 2, \dots, n$ then it also assumes integral values for all the other integral values of x.

(b) Prove that every polynomial of degree n which taken on integral values for some $n+1$ consecutive integral values of x assumes an integral value for any arbitrary integer x as well.

(c) Prove that if a polynomial $P(x)$ of the nth degree assumes integral values for $x = 0, 1, 4, 9, 16, \dots, n^2$ then it assumes an integral value for any integral value of x which is a perfect square (but such a polynomial must not necessarily assume integral values for *all* integers x).

Give an example of a polynomial which assumes an integral value for every integral value of x which is a perfect square and at the same time assumes fractional values for some other integral values of x.

10. Complex Numbers

321. **(a)** Prove that

$$\cos 5\alpha = \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha$$

and

$$\sin 5\alpha = \sin^5\alpha - 10\sin^3\alpha\cos^2\alpha + 5\sin\alpha\cos^4\alpha$$

* Leonard Euler (1707-1783), a Swiss mathematician who spent most of his life in Russia was undoubtedly one of the greatest mathematicians of the 18th century. He contributed many outstanding results to various divisions of mathematics, mechanics and physics.

(b) Prove that for natural n we have

$$\cos n\alpha = \cos^n\alpha - C(n,\ 2)\cos^{n-2}\alpha\sin^2\alpha + C(n,\ 4)\cos^{n-4}\alpha\sin^4\alpha - \\ - C(n,\ 6)\cos^{n-6}\alpha\sin^6\alpha + \dots$$

and

$$\sin n\alpha = C(n,\ 1)\cos^{n-1}\alpha\sin\alpha - C(n,\ 3)\cos^{n-3}\alpha\sin^3\alpha + \\ + C(n,\ 5)\cos^{n-5}\alpha\sin^5\alpha - \dots$$

where $C(n,\ k)$ is the number of combinations of n things, taken k at a time and the dots designate the other terms the general rule for whose construction can easily be guessed and which can be written, in succession, as long as the binomial coefficients $C(n, k)$ make sense.

Remark. Problem 321 (b) is obviously a generalization of Problem 321 (a).

322. Express $\tan 6\alpha$ in terms of $\tan\alpha$.

323. Prove that if $x + 1/x = 2\cos\alpha$ then $x^n + \frac{1}{x^n} = 2\cos n\alpha$.

324. Prove that

$$\sin\varphi + \sin(\varphi+\alpha) + \sin(\varphi+2\alpha) + \dots \\ \dots + \sin(\varphi+n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$$

and

$$\cos\varphi + \cos(\varphi+\alpha) + \cos(\varphi+2\alpha) + \dots \\ \dots + \cos(\varphi+n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\cos\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$$

325. Simplify the expressions

$$\cos^2\alpha + \cos^2 2\alpha + \dots + \cos^2 n\alpha$$

and

$$\sin^2\alpha + \sin^2 2\alpha + \dots + \sin^2 n\alpha$$

326. Simplify the expressions

$$\cos\alpha + C(n,\ 1)\cos 2\alpha + C(n,\ 2)\cos 3\alpha + \dots \\ \dots + C(n,\ n-1)\cos n\alpha + \cos(n+1)\alpha$$

and

$$\sin\alpha + C(n,\ 1)\sin 2\alpha + C(n,\ 2)\sin 3\alpha + \dots \\ \dots + C(n,\ n-1)\sin n\alpha + \sin(n+1)\alpha$$

327. Prove that if m, n and p are arbitrary integers then the expression

$$\sin\frac{m\pi}{p}\sin\frac{n\pi}{p}+\sin\frac{2m\pi}{p}\sin\frac{2n\pi}{p}+\sin\frac{3m\pi}{p}\sin\frac{3n\pi}{p}+\dots$$
$$\dots+\sin\frac{(p-1)\,m\pi}{p}\sin\frac{(p-1)\,n\pi}{p}$$

is equal to $-p/2$ when $m+n$ is divisible by $2p$ and $m-n$ is not, is equal to $p/2$ when $m-n$ is divisible by $2p$ and $m+n$ is not and is equal to zero when both $m+n$ and $m-n$ are divisible by $2p$ or not divisible by $2p$.

328. Prove that

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+2}+\cos\frac{6\pi}{2n+1}+\dots+\cos\frac{2n\pi}{2n+1}=-\frac{1}{2}$$

329. Write equations whose roots are equal to the numbers

(a) $\sin^2\frac{\pi}{2n+1},\quad \sin^2\frac{2\pi}{2n+1},\quad \sin^2\frac{3\pi}{2n+1},\quad \dots,\quad \sin^2\frac{n\pi}{2n+1};$

(b) $\cot^2\frac{\pi}{2n+1},\quad \cot^2\frac{2\pi}{2n+1},\quad \cot^2\frac{3\pi}{2n+1},\quad \dots,\quad \cot^2\frac{n\pi}{2n+1}.$

330. Simplify the expressions of the sums

(a) $\cot^2\frac{\pi}{2n+1}+\cot^2\frac{2\pi}{2n+1}+\cot^2\frac{3\pi}{2n+1}+\dots+\cot^2\frac{n\pi}{2n+1};$

(b) $\csc^2\frac{\pi}{2n+1}+\csc^2\frac{2\pi}{2n+1}+\csc^2\frac{3\pi}{2n+1}+\dots+\csc^2\frac{n\pi}{2n+1}.$

331. Simplify the expressions of the products

(a) $\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\sin\frac{3\pi}{2n+1}\dots\sin\frac{n\pi}{2n+1}$

and

$$\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\sin\frac{3\pi}{2n}\dots\sin\frac{(n-1)\,\pi}{2n}$$

(b) $\cos\frac{\pi}{2n+1}\cos\frac{2\pi}{2n+1}\cos\frac{3\pi}{2n+1}\dots\cos\frac{n\pi}{2n+1}$

and

$$\cos\frac{\pi}{2n}\cos\frac{2\pi}{2n}\cos\frac{3\pi}{2n}\dots\cos\frac{(n-1)\,\pi}{2n}$$

332. Show that from the results established in Problems 330 (a) and (b) it follows that for any positive integer n the value of the sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}$$

lies between $\left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)\frac{\pi^2}{6}$ and $\left(1-\frac{1}{2n+1}\right)\times$ $\times\left(1+\frac{1}{2n+1}\right)\frac{\pi^2}{6}$.

Remark. In particular, the assertion stated in Problem 332 implies that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

where the sum of the *infinite series* $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is understood as the limit to which the finite sum $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$ tends as $n\to\infty$.

---

333. **(a)** A point M lies on a circle circumscribed about a regular n-gon $A_1A_2\ldots A_n$. Prove that the sum of the squares of the distances from that point to all the vertices of the n-gon is independent of the position the point occupies on the circle and is equal to $2nR^2$ where R is the radius of the circle.

(b) Prove that the sum of the squares of the distances from an arbitrary point M lying in the plane of a regular n-gon $A_1A_2\ldots A_n$ to all the vertices of the n-gon is dependent solely on the distance l between M and the centre O of the n-gon and is equal to $n(R^2+l^2)$ where R is the radius of the circle circumscribed about the n-gon.

(c) Prove that the assertion stated in Problem 333 (b) remains true for the case when the point M does not lie in the plane of the n-gon $A_1A_2\ldots A_n$.

334. Let M be a point lying on an arc A_1A_n of a circle circumscribed about a regular n-gon $A_1A_2\ldots A_n$. Prove that

(a) if n is even then the sum of the squares of the distances from the point M to the vertices of the n-gon with even indices is equal to the sum of the squares of the distances from M to the vertices with odd indices;

(b) if n is odd then the sum of the distances from the point M to the vertices of the n-gon with even indices is equal to the sum of the distances from M to the vertices with odd indices.

Remark. For a geometrical proof of the theorem stated in Problem 334(b) see the solution of Problem 137 in book [8].

335. The radius of the circle circumscribed about a regular n-gon $A_1A_2\ldots A_n$ is equal to R. Prove that

(a) the sum of the squares of all sides and of the squares of all diagonals of the n-gon is equal to n^2R^2;

(b) the sum of all sides and of all diagonals of the n-gon is equal to $n\cot(\pi R/2n)$;

(c) the product of all sides and of all diagonals of the n-gon is equal to $n^{n/2}R^{n(n-1)/2}$.

336*. Find the sum of the 50th powers of all the sides and of all the diagonals of a regular 100-gon inscribed in a circle of radius R.

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337. It is known that $\left|z+\frac{1}{z}\right|=a$ where z is a complex number. What are the greatest and the least possible values of the modulus $|z|$ of the complex number z?

338. Let a sum of n complex numbers be equal to zero. Prove that among them there are two numbers whose arguments differ by not less than 120°.

Is it possible to replace in this problem the angle of 120° by a smaller angle?

339. Let $c_1, c_2, \ldots, c_n$ and z be complex numbers such that

$$\frac{1}{z-c_1}+\frac{1}{z-c_2}+\ldots+\frac{1}{z-c_n}=0$$

Prove that if the numbers $c_1, c_2, \ldots, c_n$ are represented in the complex plane by the vertices of a convex n-gon then the number z is represented by a point lying inside that n-gon.

11. Several Problems in Number Theory

340. *Fermat's theorem.* Prove that if p is a prime number then for any whole number a the difference $a^p - a$ is divisible by p.

Remark. The assertions proved in Problems 46 (a)-(e) are special cases of this theorem.

341. *Euler's theorem.* Let N be a whole number and r be the number of integers belonging to the sequence $1, 2, 3, \ldots, N-1$ which are relatively prime to N. Prove that if a is an arbitrary whole number relatively prime to N then the difference $a^r - 1$ is divisible by N.

Remark. If N is a prime number then all the numbers in the sequence 1, 2, 3, ..., $N-1$ are relatively prime to N, that is $r=N-1$. In this case Euler's theorem reduces to the following theorem: the difference $a^{N-1}-1$ where N is a prime number is divisible by N. We thus see that Fermat's theorem (Problem 340) can obviously be regarded as a special case of Euler's theorem.

If $N=p^n$ where the number p is prime then among the $N-1=p^n-1$ numbers $1, 2, 3, \ldots, N-1$ only the numbers $p, 2p, 3p, \ldots, N-p=(p^{n-1}-1)p$ are not relatively prime to $N=p^n$. In this case $r=(p^n-1)-(p^{n-1}-1)=p^n-p^{n-1}$, and hence Euler's theorem reduces to the following theorem: the difference $a^{p^n-p^{n-1}}-1$ where the number p is prime and a is not divisible by p must necessarily be divisible by p^n.

If $N=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct prime numbers then the number r of positive integers which are less than N and are relatively prime to N is given by the formula

$$r = N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_n}\right)$$

If N is a power of a prime number p, that is $N = p^n$, then this formula yields

$$r = p^n\left(1-\frac{1}{p}\right) = p^n - p^{n-1}$$

The last formula coincides with the result established above.

342*. According to Euler's theorem, the difference $2^k - 1$ where $k = 5^n - 5^{n-1}$ is divisible by 5^n (see Problem 341 and, in particular, the remark to it). Prove that the difference $2^k - 1$ cannot be divisible by 5^n for any k less than $5^n - 5^{n-1}$.

343. Let us write consecutively the powers of the number 2:

$$2;\ 4;\ 8;\ 16;\ 32;\ 64;\ 128;\ 256;\ 512;\ 1024;\ 2048;\ 4096;\ \dots$$

It can easily be noticed that the last digits of the numbers forming this sequence repeat periodically with period of length 4:

$$2;\ 4;\ 8;\ 6;\ 2;\ 4;\ 8;\ 6;\ 2;\ 4;\ 8;\ 6;\ \dots$$

Prove that, beginning with some number belonging to the sequence of the powers of 2, the last 10 digits of the numbers forming that sequence also repeat periodically. Find the length of the period and the number in the sequence beginning with which this periodicity takes place.

344*. Prove that there exists a power of the number 2 such that the last 1000 digits in its decimal representation are all ones and twos.

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345. We shall call a pair of (different) natural numbers m and n "good" if they contain the same prime factors (in the general case raised to different powers). For example, such are the numbers $90 = 2\cdot 3^2\cdot 5$ and $150 = 2\cdot 3\cdot 5^2$. Further, we shall call such a pair "very good" in case both m, n and $m+1$, $n+1$ are "good pairs" (for example, the numbers $6 = 2\cdot 3$ and $48 = 2^4\cdot 3$ form a "very good" pair because $6+1 = 7$ and $48+1 = 49 = 7^2$). Is the set of all "very good" pairs of natural numbers finite or not?

346. Let a, $a+d$, $a+2d$, $a+3d$, ... be an arbitrary (infinite) arithmetic progression whose first term a and common difference d are natural numbers. Prove that the progression contains infinitely many terms whose factorizations contain the same prime factors (of course, in the general case, the exponents of their powers contained in the factorizations may be different).

347. *Wilson's* theorem.* Prove that if p is a prime number then the number $(p-1)!+1$ is divisible by p and if p is a composite number then $(p-1)!+1$ is not divisible by p.

348. Prove that

(a) for any prime number p there are integers x and y such that x^2+y^2+1 is divisible by p;

(b*) if the division of a prime number p by 4 leaves a remainder of 1 then there exists an integer x such that x^2+1 is divisible by p (in case the prime number p is odd the condition imposed on p is necessary and sufficient for the existence of such x).

Problem 348 is related to the part of number theory studying *the representation of natural numbers as sums of powers* (with equal exponents $n>1$) *of some other natural numbers.* For example, from the result established in Problem 348 (b) we can draw the conclusion that *a natural number N can be represented in the form of a sum of squares of two natural numbers if and only if the factorization of N as a product of prime factors contains even powers of prime factors of the form $4n+3$ (that is of all prime factors whose division by 4 leaves a remainder of 3).*

From the result established in Problem 348 (a) it is possible to deduce an interesting theorem asserting that *all the natural numbers without exception can be represented as sums of squares of four natural numbers* (or of a smaller number of squares). In its turn, this theorem makes it possible to prove that *each natural number can be represented in the form of a sum of a bounded number of fourth powers of natural numbers, say as a sum of 53 (or less) exact fourth powers of natural numbers.* (More intricate methods make it possible to replace 53 by 21; the last result can probably be made more precise: some considerations indicate that every natural number can probably be represented as a sum of not more than 19 exact fourth powers of integers.) It is also proved that *every natural number can be represented as a sum of not more than nine cubes of integers* (in this case the number 9 cannot be replaced by a smaller number!).

All these assertions are special cases of the following remarkable theorem: *for any positive integer k there is an integer N* (which, of course, depends on k) *such that any positive integer can be represented in the form of a sum of not more than N summands each of which is the kth power of an integer**.* There exist several different proofs of the last theorem but in the proofs that were known until recent years extremely intricate mathematical methods (related to higher mathematics) were used. It was only in 1942 that the Soviet mathematician Yu. V. Linnik constructed a purely arithmetical proof of the theorem which however is very complicated. It was also established that *each rational number can be represented as a sum of not more than three cubes of rational numbers*; in this connection it is interesting to note that *the number 1 cannot be represented in the form of a sum of cubes of two rational numbers.*

349. Prove that there are infinitely many prime numbers.

350. (a) Prove that among the terms of the arithmetic progressions 3; 7; 11; 15; 19; 23; ... and 5; 11; 17; 23; 29; 35; ... there are infinitely many prime numbers.

* John Wilson (1714-1793), a Scottish astronomer and mathematician.

** This theorem is often referred to as *Waring's problem* after Edward Waring (1734-1798), an English mathematician who posed this problem. — *Tr.*

(b*) Prove that among the terms of the arithmetic progression

$$11;\ 21;\ 31;\ 41;\ 51;\ 61;\ \ldots$$

there are infinitely many prime numbers.

By analogy with the solutions of Problems 350 (a)-(b), but in a more complex manner, it can be proved that among the terms of the arithmetic progression 5; 9; 13; 17; 21; 25; ... there are also infinitely many prime numbers. There also holds a more general assertion: *any arithmetic progression whose first term is relatively prime to its common difference contains an infinitude of prime numbers.* This assertion is proved in an extremely difficult way. It is interesting to mention that a proof of this classical theorem of number theory in which the methods of higher mathematics are not used was for the first time elaborated in 1950 by the Danish mathematician A. Selberg (this proof is however very complicated). Before that only proofs based on higher mathematics were known.

Solutions

1. Let A be the first of the chosen soldiers and B the second one. If A and B are in one line then B is taller than A because A is the smallest soldier in his line; if A and B are in one file then B is also taller than A because B is the tallest soldier in that file. Finally, if A and B are in different lines and in different files, and if C, another soldier, is in one line with A and in one file with B, then B is taller than A since B is taller than C while A is smaller than C.

2. Let us consider the sum of the numbers of times each person has ever shaken hands. The sum must necessarily be even because when two persons A and B shake hands the number of times A has shook hands increases by 1 and the number of times B has ever shook hands also increases by 1, and hence this adds the number 2 to the total sum of the numbers of the handshakes. Since this sum consists of the numbers of times each person has shaken hands and the sum is even, it follows that the number of odd addends in this sum is even, which is what we intended to prove.

3. Let A be one of the six people. It is clear that

1° either A has three acquaintances B_1, B_2, and B_3 among the other five persons or

2° there are three persons C_1, C_2 and C_3 with neither of whom A is acquainted (because A is either acquainted with three of the five persons different from A or is not acquainted with three persons among those five people).

If case 1° takes place and among B_1, B_2 and B_3 there are not two persons who are acquainted with each other then B_1, B_2 and B_3 form the triple of persons whose existence is asserted in the problem; if in case 1° two of the three persons B_1, B_2 and B_3, say B_1 and B_2, are acquainted with each other then among the three persons A, B_1 and B_2 any two persons are acquainted with each other. Similarly, if case 2° takes place and any two of the three persons C_1, C_2 and C_3 are acquainted with each other then they form the triple we are interested in, and if in case 2° there are two persons among C_1, C_2 and C_3, say C_1 and C_2, who are not acquainted with each other then the triple consisting of A, C_1 and C_2 is the one whose existence we want to prove.

4. (a) Each of the N persons present at the meeting can have $0, 1, 2, \ldots, N-1$ acquaintances, that is the greatest possible number of his acquaintances is equal to N. However, if somebody has 0 acquaintances then nobody has $N-1$ acquaintances. On the contrary, if somebody has $N-1$ acquaintances then nobody has 0 acquaintances. It follows that there must necessarily exist two people having *the same number* of acquaintances (cf. what was said about Dirichlet's principle on page 9).

(b) We shall index the people taking part in the meeting with the numbers $1, 2, \ldots, N$ and consider the following situation: let for all the values $i = 0, 1, 2, \ldots$ where $i < N/2$ the person with the index $N-i$ be acquainted with all the people except the first i persons (this means that the Nth person is acquainted with all other people without exception, the $(N-1)$th person is acquainted with all other people except the 1st one, the $(N-2)$th person is acquainted with all other people except the 1st and the 2nd, etc.) and let all the people with the indices i such that $1 \leqslant i \leqslant (N+1)/2$ be not acquainted with one another. In this case it is obvious that for $N = 3$ the 1st and the 2nd persons are acquainted only with the 3rd one while the 3rd person has two acquaintances. Similarly, for $N = 4$ the 1st person is acquainted only with the 4th one, the 2nd person with the 4th and the 3rd, the 3rd with the 4th and the 2nd persons and the 4th person has three acquaintances. In just the same manner we can readily show that for an *odd* number $N = 2k+1$ the number n_i $(i = 1, 2, \ldots, N)$ of the people the ith person is acquainted with takes on the values $n_1 = 1$, $n_2 = 2, \ldots$ $\ldots, n_k = k$, $n_{k+1} = k$, $n_{k+2} = k+1, \ldots, n_N = N-1$ and that for an *even* number $N = 2k+2$ we have $n_1 = 1$, $n_2 = 2, \ldots$ $\ldots, n_{k+1} = k+1$, $n_{k+2} = k+1$, $n_{k+3} = k+2, \ldots, n_N = N-1$. Thus, in the case under consideration *there are not three persons having the same number of acquaintances.*

5. If all the participants of the meeting are acquainted with one another then the possibility of seating four people in the required manner is quite evident. Now let us suppose that two persons A and B are *not acquainted* with each other. Each of them has not less than n acquaintances among the other $2n-2$ participants. Since $n+n = 2n = (2n-2)+2$ we conclude that A and B have at least two mutual acquaintances C_1 and C_2, and we can seat A and B opposite each other and seat C_1 and C_2 between them.

6. Let A be a scientist having the *greatest* number n of acquaintances among the participants of the congress (there can be several such scientists and then by A is meant one of them). It is clear that $n > 0$ since we supposed that some of the participants of the congress had been acquainted with one another. All the acquaintances of A have *different* numbers of acquaintances (because A is a mutual acquaintance of any two of them); besides, none of them has more than n acquaintances. Therefore B, one of the acquaintances of A, must necessarily have *only one* acquaintance, some other of the acquaintances of A has exactly two acquaintances, a third one has three acquaintances, ..., and, finally, the last (the nth) of the acquaintances of A has, like A, n acquaintances. The existence of person B proves the assertion of the problem.

7. Let us arbitrarily choose three delegates of the congress. Among them there must be two persons knowing some one language (one of the three languages). We shall lodge them in one room. From the remaining 998 delegates of the congress we again choose three persons among whom there are two people that can be lodged in one room, and so on until there remain only four delegates A, B, C and D. If every two of them can speak with each other there are no difficulties in lodging these four people; if A and B cannot communicate with each other then both C and D can serve as their interpreters (which makes the communication in the triples A, B, C and A, B, D possible). Therefore we can, for instance, lodge C and A in one room and D and B in another room.

8. Let A be one of the participants of the conference. He can speak with each of the other 16 participants in at least one of the three languages. It is readily seen that there is a language (we shall speak of this language as the first one) among the three languages that A can speak in with not less than 6 participants. Indeed, if otherwise, A could not speak with more than $5 \cdot 3 = 15$ scientists whereas, by the condition of the problem, every two scientists can speak with each another. Further, if among these 6 scientists there are two who speak with each other in this language the assertion of the problem turns out to be true. If otherwise, these 6 participants can speak with one another using only two languages.

Now, let B be an arbitrary scientist among the 6 chosen scientists. It is clear that among the other 5 scientists there are 3 with whom B can speak in one and the same language (we shall call it the second language). Indeed, if otherwise, then among these 5 participants of the conference there would be not more than $2 \cdot 2 = 4$ persons with whom B could communicate. If among these three scientists at least two, say C and D, can speak with each other in the second language then the three scientists B, C and D can speak with one another in one language, and the assertion of the problem again turns out to be true. In case these three scientists speak with one another in the third language then it is they who form a triple of scientists whose existence we intended to prove.

9. **(a)** Let us choose one of the participants of the meeting. We shall denote him A and all the persons acquainted with him $A_1, A_2, \ldots, A_k$ respectively. It is clear that among $A_1, A_2, \ldots, A_k$

there are not two persons acquainted with each other and that any two of them, say A_i and A_j, have two mutual acquaintances, namely A and A_{ij} (here $i, j = 1, 2, \ldots, k$ and $i \neq j$). Besides, it is obvious that among the $k(k-1)/2$ participants A_{ij} there are not two persons coinciding with each other because, if otherwise, that person and person A would have not less than three mutual acquaintances. On the other hand, since every participant who is not acquainted with A and the person A himself have two mutual acquaintances (they obviously belong to the set $A_1, A_2, \ldots, A_k$), we see that *all* the participants who are not acquainted with A are $A_{12}, A_{13}, \ldots, A_{k-1, k}$, and therefore the total number n of the participants of the meeting is expressed as

$$n = 1 + k + \frac{k(k-1)}{2} \qquad (*)$$

(here 1 corresponds to A, k corresponds to all persons A_i and $k(k-1)/2$ corresponds to the persons A_{ij}).

Now we note that by virtue of equality (*) which can also be rewritten as

$$k^2 + k - (2n-2) = 0 \qquad (**)$$

it follows that

$$k = -\frac{1}{2} + \sqrt{\frac{1}{4} + (2n-2)} = \frac{\sqrt{8n-7} - 1}{2}$$

(the other root of quadratic equation (**) is $k' = -1/2 - \sqrt{1/4 + 2n - 2}$; it is negative and must therefore be discarded). Hence, the number k of the people who are acquainted with an arbitrarily chosen person A is uniquely determined by the total number n of the participants of the meeting, that is k is *one and the same* for *all* persons A.

(b) By (*), we have

$$n = \frac{k(k+1)}{2} + 1 \qquad (***)$$

whence it follows that n exceeds by unity the number $k(k+1)/2$, the latter being one of the so-called *triangular numbers* for n expressed by formula (***) the number of the acquaintances each of the participants has is equal to k; here $k = 1, 2, 3, \ldots$ is an arbitrary natural number.

10. Let A, B and C be three arbitrary inhabitants of the town. It is evident that *there can be the case when all the three people are friends*; *it is also possible that one of them* (*say A*) *is neither a friend of B nor of C while B and C are friends.* Then for A, B and C to make friends with one another it is sufficient that A should quarrel with all his friends and make friends with all his enemies. It can also be easily seen that the other two cases when *all the three inhabitants A, B and C are enemies* and when *one of the inhabitants, say A, is a friend of both B and C while B and C are enemies* are impossible. Indeed, in both cases among the three pairs A, B; A, C and B, C of the inhabitants of the town of Manifold there is an odd number c (equal to 3 or 1) of pairs of enemies and an even number e (equal to 0 or 2) of pairs of friends. In all the cases when A or B or C quarrels with his friends and makes friends with his enemies the odd number c and the even number e either do not change or are replaced by an odd number c' and an even number e' respectively, whence it follows that all the three persons A, B and C can never make friends with one another (because the number c cannot become equal to 0).

The description of the "friendship relations" between any three persons A, B and C shows that for the whole population of the town these relations can be described in the following way: there are two groups of people in the town (*two parties $\mathscr{M}$ and $\mathscr{N}$*) *such that each of the inhabitants of the town belongs either to one party or to the other* (but never to both parties simultaneously), *every two of the members of one party being friends and any two inhabitants belonging to the different parties being enemies.* Indeed, let us add to the above three inhabitants A, B and C another inhabitant D of the town of the Manifold. If A and B are friends and D is a friend of at least one of them, then D is also a friend of the other and hence he belongs to the same party as A and B; if A and B are enemies then D is a friend of only one of them (and must necessarily be a friend of one of them). This argument shows that it is possible to divide the *four-tuple* of the inhabitants A, B, C and D into two parties $\mathscr{M}$ and $\mathscr{N}$ (however, one of the parties may turn out to be "void": this is the case when all the inhabitants A, B, C and D are friends). Proceeding in this way, that is adding consecutively new persons to the ones we have already considered, we prove the possibility of dividing *all* the 10 000 inhabitants of the town into the two parties $\mathscr{M}$ and $\mathscr{N}$.

Now we can readily prove the assertion stated in the problem. If all the inhabitants of the town are friends then no proof is needed. If neither of the parties $\mathscr{M}$ and $\mathscr{N}$ is "void" then it is sufficient that every day one of the members of party $\mathscr{M}$ should leave $\mathscr{M}$ and join the other party $\mathscr{N}$. If the number of the members of party $\mathscr{M}$ is k then all the inhabitants of the town can become friends in k days. It follows that the period of 5000 days (5000 days $\approx$ 14 years) is sufficient for all the inhabitants of the town to become friends (because at least one of the parties $\mathscr{M}$ and $\mathscr{N}$ consists of not more than 5000 people).

11. It is natural to consider a line segment joining two points representing two castles as a "road" connecting these castles (see Fig. 4). All the castles in the state of Oz are connected by a finite number n of roads. If the knight travels in the country sufficiently long he goes along sufficiently many roads. If the number N of these roads is not less than $4n+1$ then the knight must go along at least one road AB (where A and B are the castles connected by that road) not less than 5 times. Besides, not less than three times he must go along this road in *one and the same direction* (say, from A to B). Therefore if BC and BD are the other two roads starting from castle B then the knight must at least twice turn in one and the same direction when he leaves B (where both times he came from A), say when he leaves B the

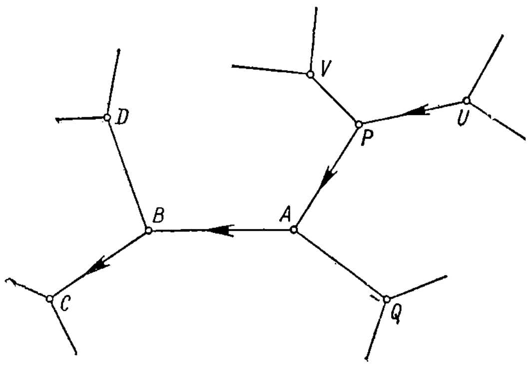


Fig. 4

ith and the jth time, $j > i$, and goes, say, from B to C. But in that case the conditions of the problem imply that the knight not only comes to B from one castle (from castle A) the ith and the jth time but he also comes to A both times from one and the same castle (in Fig. 4 the castle from which the knight both times comes to A is denoted as P; for, if the knight turns to road BC when he leaves B, that is turns to the left, then he must turn to the right when he leaves A, which means that he comes to A from P). We can similarly prove that *the routes of the knight which brought him to B the ith and the jth time coincide completely*; for instance, he comes to castle P both times from one and the same castle (which is denoted as U in Fig. 4), etc. It follows that if before his ith visit to B the knight goes past a number k of castles after he has left his own castle X then he must necessarily be again at X before he goes past k castles to visit B the jth time, which concludes the proof of the assertion stated in the problem.

12. Let us agree to call "friends" any two knights who are not enemies. Further, we begin with seating all the knights at the round table *in an arbitrary way.* Suppose that it turns out that knight A sits next to his enemy B. For definiteness, we suppose that B sits to the *right* of A. Now we shall prove that *there are two seats where two knights A', a friend of A, and B', a friend of B, sit next to each other, the knight B' sitting to the right of A'* (see Fig. 5*a*). Indeed, A has not less than n friends. The number of seats to the right of the n friends of A is also equal to n, and the number of the enemies of B does not exceed $n-1$, whence it follows that there is at least one of the seats to the right of the knight A', a friend of A, where knight B', a friend of

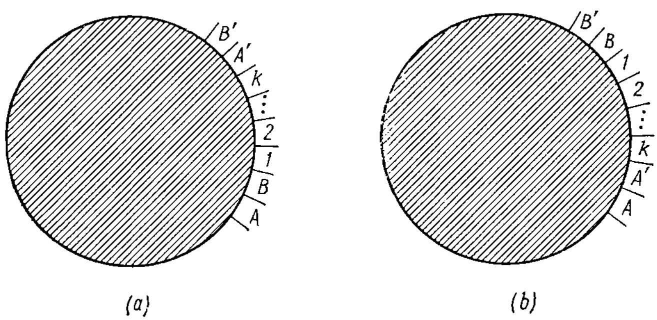


Fig. 5

knight B, sits. Now, *let all the knights from B to A' inclusive who sit to the right of A change their seats and sit in the reverse order* (see Fig. 5*b*). This will obviously change only the pairs A, B and A', B' of knights sitting next to each other: they will be replaced by the pairs of friends A, A' and B, B' respectively. This means that the number of pairs of enemies sitting next to each other will decrease by at least 1 (it will even decrease by 2 in case knights A' and B' are enemies). Therefore, if Merlin continues to make the knights change their seats in the same manner all the pairs of enemies sitting next to each other will eventually be separated.

13. (a) Let us divide the given coins into three groups so that in each of the first two groups there are 27 coins and the third group contains 26 coins. In the first weighing let us put on the scale pans the groups of 27 coins. If these groups do not balance then the false coin is in the lighter group and if these groups balance then the false coin is in the group of 26 coins. We see that Problem 13 (a) reduces to the following problem: given 27 coins among which there is a false one, it is required to detect the false coin by means of three weighings, for the problem of detecting a false coin in the group of 26 coins can be reduced to the former by adding to those 26 coins one more arbitrary coin taken from the remaining 54 coins.

In the second weighing we take the group of 27 coins containing the false one and divide it into three groups of 9 coins each. On putting two of these groups of 9 coins on the scale pans we find the group of 9 coins containing the false coin.

Next we take the group of 9 coins containing the false one, divide it into three groups of 3 coins each and find the triple of coins containing the false coin.

Finally, proceeding in the same manner we detect the false coin by means of the fourth weighing.

(b) Let k be a natural number for which the inequalities $3^{k-1} < n \leqslant 3^k$ hold. We shall show that the number k satisfies the conditions of the problem.

To begin with, we shall prove that it is always possible to detect the false coin by means of k weighings. Let us divide the n given coins into three groups so that the first two groups contain 3^{k-1} (or less) coins each, the third group containing not more than 3^{k-1} coins (this is possible since $n \leqslant 3^k$). On putting the first two groups on the scale pans we find which of the three groups contains the false coin (cf. the solution of Problem 13a). Hence, after the first weighing we find a group of 3^{k-1} coins containing the false one (in case the false coin is in the group containing less than 3^{k-1} coins we add to these coins the required number of arbitrary coins so that the resultant number of coins is equal to 3^{k-1}). In every consecutive weighing we divide the group of coins containing the false coin (that is the group which was found in the preceding weighing) into three groups of equal number of coins in the way described above and find the smaller group containing the false coin. Hence, after k weighings we arrive at a group of one coin, that is we detect the false coin.

Now it remains to show that k is the *minimum* number of weighings with the aid of which it is always possible to detect the false coin, that is it remains to show that after $k-1$ weighings performed in an arbitrary manner there may occur an unfavourable case when the false coin is not detected.

In every weighing the remaining coins are divided into three groups, namely the two groups which are then put on the scale pans and the third group which is not put on either of the pans. If the groups put on the scale pans contain the same number of coins and if they balance each other then the false coin must be in the group which is not put on either of the pans in this weighing. If one of the groups put on the scale pans turns out to be heavier and the number of coins in each of these groups is the same, then the false coin is in the lighter group. Finally, if we put different numbers of coins on the scale pans then in the case when the group containing the greater number of coins turns out to be heavier the false coin may be in any of the three groups and hence such a weighing gives us no information about the group containing the false coin. Now suppose that in a sequence of arbitrary weighings the result of each weighing turns out to be most unfavourable, that is every time the false coin is in the group which contains the greater number of coins. Then after each weighing the number of coins in the group containing the false coin decreases not more than 3 times (because when a group of a certain number of coins is divided into three groups at least one of the smaller groups always contains not less than one third of the number of coins in the former group). Consequently, after $k-1$ weighings the number of coins in the group containing the false coin remains not less than $n/3^{k-1}$, and since $n > 3^{k-1}$ the false coin is not detected after $k-1$ weighings.

Remark. The answer to Problem 13 (b) can priefly be stated in the following way: the minimum number of weighings required for detecting the false coin in a group of n coins is equal to $[\log_3(n-1/2)]+1$ where the square brackets denote the *integral part* of a number (see page 36).

14. Let us choose two arbitrary cubes and put them on the scale pans (the first weighing). Here there can be the following two different cases.

1°. *In the first weighing one of the cubes turns out to be heavier.* Then one of the two cubes we are weighing must necessarily be made of aluminium and the other of duralumin. Next we put these two cubes on one of the scale pans and on the other scale pan we put, in succession, each of the 9 pairs of cubes into which the 18 remaining cubes are divided in an arbitrary way. If one of these pairs turns out to be heavier than the initial pair, that means that both cubes in the new pair are made of duralumin; if the initial pair of cubes is heavier then both cubes in the new pair are made of aluminium. Finally, if the two pairs balance, the new pair consists of an aluminium cube and a duralumin cube. Thus, in case 1° the number of cubes made of duralumin can be determined with the aid of 10 weighings (because after the first weighing we perform 9 more weighings).

2°. *In the first weighing the first two cubes balance.* Then the cubes of the first pair are either both made of aluminium or both made of duralumin. Next we put these two cubes on one scale pan and on the other scale pan we put, in succession, each of the 9 pairs of cubes into which the remaining 18 cubes are arbitrarily divided. Let us suppose that the first k of these pairs turn out to have the same weight as the initial pair while the $(k+1)$th pair is of some other weight. (If $k=9$ then all the cubes are of the same weight and, consequently, there are no duralumin cubes among them at all; the case when $k = 0$ does not differ from the general case.) For definiteness, let us suppose that the $(k+1)$th pair turns out to be heavier than the initial pair (the argument remains almost the same when the $(k+1)$th pair turns out to be lighter). Then the first two cubes and, consequently, the cubes forming the k pairs which have the same weight as the first one must necessarily be made of aluminium. Hence, on performing $1+(k+1) = k+2$ weighings we find $k+1$ pairs of aluminium cubes. Next we put on the scale pans the two cubes forming the last pair we have weighed (this is the $(k+3)$th weighing). If both cubes turn out to be of one weight then they both are made of duralumin; if otherwise, one of them is made of aluminium and the other is made of duralumin. In both cases after $(k+3)$ weighings we can find a pair of cubes one of which is made of aluminium while the other is made of duralumin. Using this pair we can perform $8-k$ weighings to determine the number of duralumin cubes among the remaining $20-2(k+2) = 16-2k$ cubes by analogy with what we did in case 1°. We see that in case 2° the total number of weighings is equal to $k+3+(8-k) = 11$.

15. Let us divide the given coins into three groups of four coins each. In the first weighing on each of the scale pans we put a group of four coins. There can be the following two cases here:

1°. The two groups balance.

2°. One of the groups turns out to be heavier.

Let us consider separately each of these possibilities.

1°. *In the first weighing the two groups of four coins balance.* This means that the false coin is in the third group while the 8 coins put on the scale pans are genuine. Let us index the coins in the remaining (third) group with the numbers 1, 2, 3 and 4. In the second weighing we put coins 1, 2 and 3 on one scale pan and three of the 8 coins known to be genuine on the other pan. Here the following two sub-cases are possible:

A. The two groups of 3 coins put on the scale pans balance. Then coin 4 is false. On weighing this coin and a genuine one we find whether the false coin is lighter or heavier than the genuine coin.

B. One of the two groups of 3 coins turns out to be heavier. In this case one of the coins 1, 2 and 3 is false. If the group of three genuine coins turns out to be heavier then the false coin is lighter than a genuine coin. With the aid of one more weighing we easily find the lighter of the three coins 1, 2 and 3 (cf. the solution of Problem 13a). If the group of coins 1, 2 and 3 is heavier then the false coin is heavier than a genuine coin. In that case as well we readily detect it by means of one more weighing.

2°. *In the first weighing one of the groups of four coins turns out to be heavier.* Then all the coins in the remaining third group are genuine. Let us index the four coins in the heavier group as 1, 2, 3 and 4 (if one of these coins is false then it is *heavier* than a genuine coin) and let us denote the four coins on the other scale pan as 1′, 2′, 3′ and 4′ (if one of the coins in the latter group is false then it is *lighter* than a genuine coin). In the second weighing we put coins 1, 2 and 1′ on one scale pan and coins 3, 4 and 2′ on the other pan. Here we can have the following three possible sub-cases:

A. The groups of 3 coins put on the scale pans balance. Then one of the coins 3′ and 4′ is false (and it is lighter than a genuine coin). In the third weighing we put coin 3′ on one scale pan and coin 4′ on the other scale pan. The coin that turns out to be lighter in this weighing is the false one.

B. The group of coins 1, 2 and 1′ turns out to be heavier. In this case coins 3, 4 and 1′ are genuine, for if one of the coins 3 and 4 were heavier than the others or if coin 1′ were lighter than the others, the group of coins 3, 4 and 2′ would be heavier in the second weighing, which is not the case. Thus, either one of coins 1 and 2 is false (and then the false coin is heavier than a genuine coin) or coin 2′ is false (in the latter case the false coin is lighter than a genuine coin). Let us put, in the third weighing, coin 1 on the scale pan and coin 2 on the other. If these coins balance then coin 2′ is false and if one of the two coins is heavier then it is this heavier coin that is false.

C. The group of coins 3, 4 and 2′ turns out to be heavier. Then arguing by analogy with the above we conclude that coins 1, 2 and 2′ are genuine and that either one of the coins 3 and 4 is false and is heavier than a genuine coin or coin 1′ is false and is lighter than a genuine coin. In the third weighing we put coin 3 on one scale pan and coin 4 on the other. If these coins balance then it is coin 1′ that is false. If one of the two coins turns out to be heavier then it is the heavier coin that is false.

16. (a) It is sufficient to cut the third link; then the chain is divided into two parts consisting of 2 and 4 links respectively and one separate link that was cut. On the first day the man gives the innkeeper the link that was cut; on the second day he takes this link back from the innkeeper and gives him in exchange the part of the chain consisting of two links; on the third day he again gives the innkeeper the link that was cut in addition to the links already given; on the fourth day he takes back all the links he gave before and gives the innkeeper the part of the chain consisting of four links; on the fifth day he once again gives the innkeeper the link that was cut; on the sixth day he takes that link back and gives the innkeeper the part of the chain consisting of two links in exchange; finally, on the seventh day he gives the innkeeper the remaining link.

(b) We begin with solving the following problem: what must be the greatest number n of an n-link chain for which it is sufficient to cut k links so that it is possible to take any number of links from 1 to n inclusive using some (or all) of the parts into which the chain is divided? To solve this problem let us find what is the best arrangement of the k links to be cut. After k links have been cut we have these k separate links at our disposal and therefore any number of links ranging from 1 to k inclusive can be taken by using these k links only. But it is impossible to take $k+1$ links if we do not have a part of the chain consisting of $k+1$ links or of a smaller number of links. Clearly, it is best to have a part consisting of exactly $k+1$ links; in this case we can use this part and the k links we have cut to have any number of links from 1 to $2k+1$. To take $2k+2=2(k+1)$ links as well it is necessary to have a part of the chain consisting of $2(k+1)$ links or of a smaller number of links. Clearly, the best case is when this part consists of exactly $2(k+1)$ links. Now we can take any number of links from 1 to $2k+1+2(k+1)=4k+3$. The next part of the chain we need must obviously consist of $4(k+1)$ links. Continuing to argue in this way we readily show that the best case is when the $k+1$ parts of the chain obtained after k links have been cut (here, when speaking of the parts of the chain, we do not regard as parts the k separate links that have been cut) consist of the following numbers of links respectively:

$$k+1, \quad 2(k+1), \quad 4(k+1), \quad 8(k+1), \ldots, \; 2^k(k+1)$$

In this case we can take any number of links from 1 to n inclusive where

$$n=k+\{k+1+2(k+1)+4(k+1)+\ldots+2^k(k+1)\}=$$
$$=k+\left(2^{k+1}-1\right)(k+1)=2^{k+1}(k+1)-1$$

by using the parts of the chain and the separate links that have been cut.

Thus, if $2^k k \leqslant n \leqslant 2^{k+1}(k+1)-1$, it is sufficient to cut k links but it is insufficient to cut $k-1$ links. In particular,

$$\begin{array}{lll} k=1 & \text{for} & 2\leqslant n\leqslant 7 \\ k=2 & \text{for} & 8\leqslant n\leqslant 23 \\ k=3 & \text{for} & 24\leqslant n\leqslant 63 \\ k=4 & \text{for} & 64\leqslant n\leqslant 159 \\ k=5 & \text{for} & 160\leqslant n\leqslant 383 \\ k=6 & \text{for} & 384\leqslant n\leqslant 895 \\ k=7 & \text{for} & 896\leqslant n\leqslant 2047 \end{array}$$

We see that for $n = 2000$ the smallest number of links that should be cut is equal to 7. The conditions of the problem will be fulfilled if we choose these links so that the 8 parts obtained after the 7 links have been cut (here the 7 separate links that have been cut are not included into the number of the parts) consist of 8, 16, 32, 64, 128, 256, 512 and 977 links respectively.

17. Let S be an arbitrary underground station and T be *the farthest station from* S in the sense that on the *shortest* way from S to T there are more (or at least not fewer) intermediate stations than on the shortest way from S to any other station. Now suppose that station T is closed. Then we can again go from S to any other station U (which is not closed) since the *shortest* way from S to U cannot go through T because, if otherwise, station U would be farther from S than station T. Therefore if U and V are two arbitrary underground stations different from T, then from one of them we can undoubtedly go to the other without passing through T. Indeed, if, for instance, U and V differ from S, then to go from U to V it is sufficient to go from U to S and then from S to V.

18. We shall prove the assertion of the problem using the *method of mathematical induction.* Let us consider the Zurbagan cross-roads from which more than two roads start. If there are only two cross-roads A and B in Zurbagan then the assertion of the problem is obvious: there are not less than two roads connecting A and B (if there were only one such road and if one-way traffic were introduced, say, in the direction from A to B, we would not be able to go from B to A). Consequently, on introducing one-way traffic from A to B in one of the roads and from B to A in the other road we can go from any cross-roads to any other cross-roads different from the former. Fortunately, this simple argument turns out to be applicable to the general situation as well. Let us suppose that *the assertion stated in the problem has already been proved for all the towns where the number of the cross-roads does not exceed n.* Let us consider another town (let this town be Zurbagan) having $n+1$ cross-roads. Let us consider two neighbouring cross-roads A and B (among these $n+1$ cross-roads) which are connected by a road AB. Suppose that one-way traffic is introduced in road AB (during the repairs), say, in the direction from A to B. Since it remains possible to go from B to A, it clearly follows that there is a "chain" of roads which does not include AB and leads from B to A. (We can assume that this "chain" of roads has no self-intersections because if the chain went twice through one and the same cross-roads C, the "cycle" of streets starting at C and returning to C again could simply be discarded.) Hence, there is a "ring" s, that is a closed network of streets in Zurbagan, which starts from A, leads to B, goes through a number of "intermediate" cross-roads and then leads to A again. Now let us consider a map of a conditional town which is obtained from the map of Zurbagan by "sticking together" all the cross-roads of ring s to form one cross-road S. All the streets which start from S in the conditional town go through the cross-roads belonging to ring s in the real town of Zurbagan *. The number of cross-roads in such a conditional town is less than $n+1$; therefore, by the hypothesis, it is possible to introduce one-way traffic in the streets of this town so that all the conditions of the problem are satisfied. Now it becomes clear that if we introduce one-way traffic (in any direction!) in the streets included in ring s and leave unchanged one-way traffic that has been introduced in the conditional "town" in all the streets of Zurbagan which are not contained in ring s, then one-way traffic will be introduced in all the streets of Zurbagan so that it will be possible to go from any of the cross-roads to any other.

19. It is clear that if there are two towns in the state of Delphinia which are connected by only one road with one-way traffic then it is impossible to get from one of these towns to the other. If there are four towns we can represent them as four vertices of a quadrilateral A_1, A_2, A_3, A_4 (see Fig. 6). It is evident that either the movement along the sides of the quadrilateral is "cyclic" (as shown in Fig. 6*a*) or there is a vertex, say A_1, such that the two sides of the quadrilateral emanating from it correspond to two roads with the traffic in the direction *from* the town represented by that vertex (see Fig. 6*b*). In the former case the vertices of the quadrilateral are quite equivalent, and any choice of the directions of movement along the diagonals of the quadrilateral does not, in principle, differ from any other choice; however, if the situation is as shown in Fig. 6*a* it is impossible to get from A_3 to A_2 going only through one "intermediate" town. In case the movement along roads A_1A_2 and A_1A_4 is in the direction from A_1, then, in accordance with the requirements stated in the problem, to go from A_2 and from A_4 to A_1 it is necessary to choose the directions of traffic along roads A_2A_3, A_4A_3 and A_3A_1 as is indicated by arrows in Fig. 6*b*. But this again leads to a "symmetric" situation for which it is sufficient to consider the organization of traffic corresponding to any (quite arbitrary!) choice of the direction of traffic along road A_2A_4, and in the case represented in Fig. 6*b* it is impossible to get from A_4 to A_2 going only through one town.

* This "map" of the conditional town can simply be understood as a table in which all the streets and all the cross-roads are enumerated and where it is indicated which streets lead to each of the cross-roads. It may happen that such a map of the "conditional town" cannot be depicted on a plane sheet of paper and that for this aim a sphere or some other more complicated surface is needed. For the argument we use here this is of no importance.

For $n = 3$ and $n = 6$ the choice of the directions of movement satisfying all the requirements imposed in the problem is possible (see Fig. 7*a* and *b*; in Fig. 7*b* even not all the roads are depicted because the situation is quite clear).

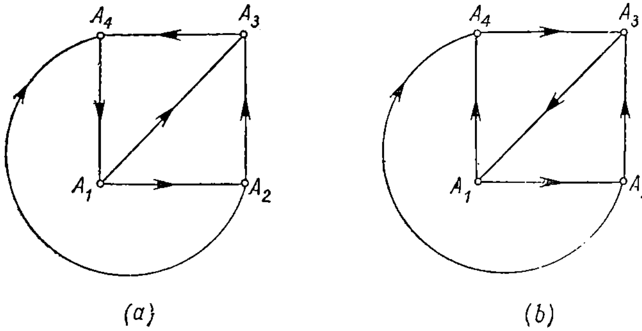


Fig. 6

Now we shall make use of the *method of mathematical induction.* Let us assume that *the assertion stated in the problem has already been proved for a number n of towns* and show that *under*

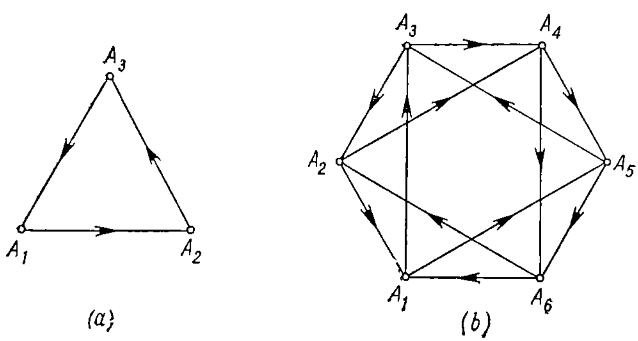


Fig. 7

this assumption the assertion must also be true for the number $n + 2$ *of towns exceeding the former number by* 2. To this end let us introduce the directions of traffic along all the roads connecting any two of the first n towns $A_1, A_2, \ldots, A_n$ in such a way that it is possible to get from any of these towns to any other going through not more than one intermediate town (according to the hypothesis, such a choice is possible). Further, along all the roads connecting the $(n+1)$th town A_{n+1} with towns $A_1, A_2, \ldots, A_n$ we introduce the directions of traffic from A_{n+1} to

$A_1, A_2, \ldots, A_n$ and along the roads connecting each of the towns $A_1, A_2, \ldots, A_n$ with the $(n+2)$th town we introduce the directions of traffic from $A_1, A_2, \ldots, A_n$ to A_{n+2} (see Fig. 8). This makes it possible to go from A_{n+1} to each of the towns $A_1, A_2, \ldots, A_n$ and from each of the towns $A_1, A_2, \ldots, A_n$ to A_{n+2} without going through any intermediate town. Finally, let the traffic along the roads connecting A_{n+1} and A_{n+2} be in the direction from A_{n+2} to A_{n+1}. Then it is possible to go from A_{n+2} to A_{n+1} without going through an intermediate town and to go from A_{n+2} to any of towns $A_1, A_2, \ldots, A_n$ and from any ot towns $A_1, A_2, \ldots, A_n$ to A_{n+1} going through one intermediate town (A_{n+1} and A_{n+2} respectively).

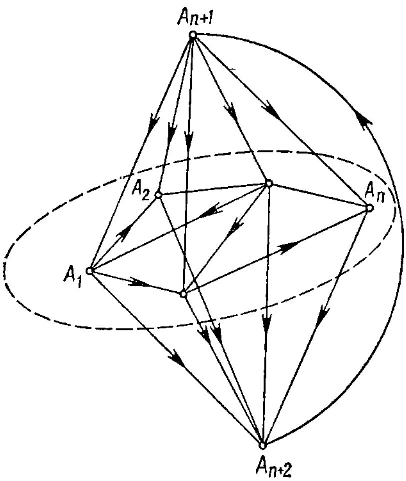


Fig. 8

Since the assertion of the problem is true for $n = 3$ and for $n = 6$ it follows that it is also true for all odd $n \geqslant 3$ and for all even $n \geqslant 6$.

20. A purely mathematical statement of the problem reads: there are 100 points (towns) in the plane (a map of the state of Shvambrania), every two points are connected either by a continuous line (which means that there is direct telephone communication between the corresponding towns) or by a dotted line (which means that there is an air route connecting the towns). Besides, *it is known that from any of the given points* (from any of the towns) *it is possible to get to any other point by tracing a chain of continuous lines* (connecting the points) *or a chain of dotted lines.* We have to prove that *among the* 100 *given points there are four points such that from any of these four points it is possible to get to any of the other three points by tracing a chain of continuous lines or a chain of dotted lines using only the lines connecting these four points.*

We shall prove the assertion *by contradiction.* To this end we shall assume that there is no such four-tuple of points and then show that, under this assumption, it is possible to choose an infinite sequence of different points (representing the corresponding towns) from the given 100 points, which of course cannot be true.

We start with two arbitrary points A_1 and A_2 connected by a *continuous* line. By the condition of the problem, A_1 and A_2 can also be connected by a chain of *dotted* lines. Let $A_1A_3A_3'A_3'' \ldots A_2$ be the *shortest* of such chains, that is the one passing through the smallest number of intermediate points (towns). Then any two points belonging to this chain which are not next to each other are connected by a *continuous* line because, if otherwise, we could shorten the chain by discarding all the points between those two points. It follows that if the chain contained more than one intermediate point (town), say points A_3, A_3' and A_3'' (where A_3'' may coincide with A_2) then the points A_1, A_3, A_3' and A_3'' (see Fig. 9*a*) would form the required four-tuple of towns (points), which contradicts the hypothesis. Therefore A_1 and A_2 are connected by the two-link chain $A_1A_3A_2$ of dotted lines.

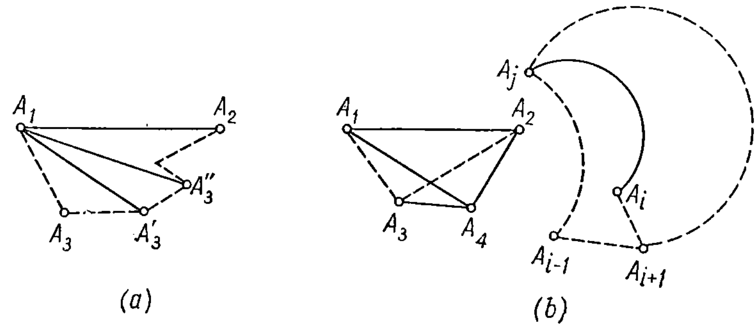


Fig. 9

Now let us consider the points A_2 and A_3 (which are connected by a dotted line). In exactly the same way as above we conclude that there exists a two-link *continuous* line $A_2A_4A_3$ which connects A_2 and A_3. Let us prove that *the point A_4 is connected with A_1 by a continuous line.* Indeed, if A_4 and A_1 were connected by a dotted line then A_1, A_2, A_3 and A_4 would form a four-tuple of points possessing the required properties, which again contradicts the hypothesis (cf. Fig. 9*b*).

Thus, we have found 4 points A_1, A_2, A_3 and A_4 such that A_2 is connected with A_1 by a continuous line, A_3 is connected with A_1 and A_2 by dotted lines and A_4 is connected with A_1, A_2 and A_3 by continuous lines. Next we shall use the *method of mathematical induction.* To this end let us assume that we have already found points $A_1, A_2, A_3, \ldots, A_i$ such that each of the points $A_3, A_5, \ldots$ (having *odd* indices) is connected with all the preceding points by *dotted* lines and each of the points $A_2, A_4, \ldots$ (having *even* indices) is connected with the preceding points by *continuous* lines and show that under this assumption the sequence of the points $A_1, \ldots, A_i$ can be continued. For definiteness, let $A_{i-1}A_i$

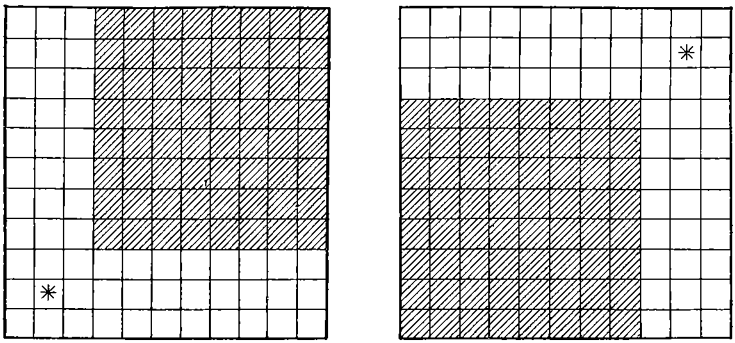
be a *continuous* line. Let us consider the *shortest broken* line $A_{i-1}A_{i+1}A_i$ connecting A_{i-1} with A_i. It is clear that the point A_{i+1} does not coincide with any of the points $A_1, A_2, \ldots, A_{i-2}$ (and, of course, with A_{i-1} and A_i either) because each of the points preceding A_{i-1} is connected by a continuous line with one of the points A_{i-1} and A_i and by a dotted line with the other point while A_{i+1} is connected by dotted lines with both of them. On the other hand, the point A_{i+1} is connected by *dotted* lines with the points $A_1, A_2, \ldots, A_{i-2}$ (and with A_{i-1} and A_i as well) because if the line A_jA_{i+1} (where $j < i-1$) is continuous then A_j, A_{i-1}, A_i and A_{i+1} form a four-tuple of points which, according to the hypothesis, cannot exist (Fig. 9*b*).

We have thus added one more point to the sequence of points (towns) we are constructing and, consequently, the sequence $A_1, A_2, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots$ can be made *infinitely long*.

21. In order to move from the left lower corner to the right upper corner of a 64-square chess-board and to pass exactly once through each of the squares of the chess-board the knight must make 63 moves. In each move the knight passes from a white square to a black one or from a black square to a white one. Therefore, after an even number of moves the knight gets to a square having the same colour as the initial square and after an odd number of moves to a square of the opposite colour. Consequently, on making 63 moves the knight cannot get to a square lying on the same diagonal as the initial square because all the squares on one diagonal are of the same colour.

22. The king can choose the following "suicidal strategy": it first moves to the left lower corner and then moves along the diagonal connecting that corner with the right upper corner. After the first move along the diagonal the king gets to the square marked by the star in Fig. 10*a*. If after the black's move following the last move of the king at least one of the rooks is outside the square of dimension 997×997 shaded in the figure, then on making its next move the king can reach a square where it must be taken. It can similarly be shown that after the 998 moves along the diagonal when the king reaches the square marked by the star in Fig. 10*b* all the rooks must be within the square of dimension 997×997 shaded in that figure. If during the king's movement at least one of the rooks remains in the same rank or in the same file as before then the king crosses that rank or that file in its movement and thus can be taken. Therefore, if the black do not want to take the king, then during the movement of the king from the position in Fig. 10*a* to the position in Fig. 10*b* (the king makes 997 moves during that period) each of the 499 rooks must make at least two moves (in every move a rook changes either its rank or its file but it cannot change both of them simultaneously). Hence, since $2 \cdot 499 = 998 > 997$, the black cannot prevent the king from being taken!

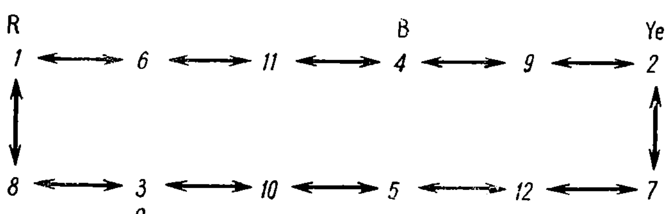
23. Let us change the order in which the squares are arranged, namely, let us rearrange them so that it becomes possible to move from any square to the neighbouring ones. In other words,



(a) (b)

Fig. 10

let us place square *6* after square *1* (because, according to the conditions of the problem, we are allowed to move a counter from square *1* to square *6*), square *11* after square *6* (because we are allowed to move a counter from square *6* to square *11*), square *4* after square *11* (because we are allowed to move a counter from square *11* to square *4*) and so on. Then we arrive at the arrangement of the squares shown in the following scheme:



We can assume that we have the 12 squares arranged in that very order and indexed as shown in the scheme (because the position which a given square occupies is in fact of no importance). We can also assume that the initial positions of the counters are as shown in the scheme where *R*, *B*, *Ye* and *G* denote the red, the blue, the yellow and the green counter respectively. For the new arrangement of the squares the rules according to which the counters move become quite simple: every counter can move to the next square on its right or on its left provided that this square is not occupied.

Now it clearly follows that the only way in which the counters can interchange their places is to move along the chain of squares in one or in the other direction: none of the counters can "outstrip" any other because the latter blocks the way. Hence, if counter *R* occupies square *4* then counter *B* must occupy square *2*, counter *Ye* must occupy square *3* and counter *G* must occupy square *1*. If counter *R* occupies square *2* then counter *B* must occupy square *3*, counter *Ye* must occupy square *1* and counter *G* must occupy square *4*. If counter *R* occupies square *3* then counter *B* must occupy square *1*, counter *Ye* must occupy square *4* and counter *G* must occupy square *2*.

No other new arrangements of the counters are possible.

24. First of all, let us prove the following auxiliary assertion: *if there are a number of students exactly n of whom* ($n \geqslant 2$) *speak English* (we symbolize this language as *e*), *exactly n speak French* (*f*) *and exactly n speak German* (*g*) *then it is possible to form a group of students in which there will be exactly 2 people speaking English, 2 people speaking French and 2 people speaking German.* It is clear that this auxiliary assertion implies the assertion stated in the problem: on forming this group of students we see that among the remaining students there are exactly $50 - 2 = 48$ people speaking English, 48 people speaking French and 48 people speaking German. From these remaining students we can again choose a group in which there are exactly 2 people speaking English, 2 people speaking French and 2 people speaking German, and among the remaining students there will be exactly $48 - 2 = 46$ people speaking English, 46 people speaking French and 46 people speaking German and so on. Then we combine five of the smaller groups thus chosen and obtain the first group in which exactly 10 people speak English, 10 people speak French and 10 people speak German. Further, in just the same way we choose smaller groups in each of which exactly 2 people speak English, 2 people speak French and 2 people speak German and then combine them to form the second and the third groups satisfying the required conditions (each of them is a combination of 5 smaller groups).

To prove the auxiliary assertion we can make use of the *method of mathematical induction.* Indeed, it is clear that for $n = 2$ the auxiliary assertion is true (in this case the group in question consists of *all* the students). Now let us assume that it is true for any number of students smaller than a value $n > 2$ and prove that under this assumption it is also true for the number of students equal to *n*. Let us denote the number of students who can speak only English as N_e, the number of students who can speak English and French but not German as N_{ef} and so on. Further, let the symbol e (or e' or e'') denote a student who can speak only English, the sumbol ef (or $(ef)'$) denote a student who can speak English and French but not German, etc. If $N_e \neq 0$, $N_f \neq 0$, $N_g \neq 0$ then we exclude three students e, f and g from the student body and thus arrive at a group of students for which, by the hypothesis, the auxiliary assertion is true. Similarly, if $N_{efg} \neq 0$ we exclude a student efg and again obtain a group of students for which the assertion is true. Further, if $N_{ef} \neq 0$, $N_{eg} \neq 0$ and $N_{fg} \neq 0$ then the assertion must also be true because three students ef, eg and fg form a group whose existence must be proved. Finally, if two of the three numbers N_{ef}, N_{eg} and N_{fg} are different from zero while the third one, say N_{fg}, is equal to zero then the numbers N_f and N_g are different from zero (because in the group of all the students ef, $(ef)'$ etc. and of all the students eg, $(eg)'$ etc. the number of the students who can speak language e exceeds the number of the students who can speak language f and the number of the students who speak language g). Therefore we can exclude two students ef and g and again arrive at a group of students in which there are exactly $n-1$ people speaking English, $n-1$ people speaking French and $n-1$ people speaking German, and, according to the hypothesis, it is possible to choose a subgroup of students from this group which satisfies the necessary requirements. Similarly, if, for instance, only N_{ef} is different from zero while $N_{eg} = N_{fg} = 0$ then obviously $N_g \neq 0$, and we can again exclude students ef and g and use the induction hypothesis. (It should be noted that the equalities $N_{ef} = N_{eg} = N_{fg} = 0$ (and $N_{efg} = 0$) contradict the assumption that at least one of the numbers N_e, N_f and N_g is equal to zero.)

Remark. It is clear that the numbers 10 and 50 in the condition of the problem are arbitrary: we can similarly prove that if in a student body there are exactly n people speaking English, n people speaking French and n people speaking German then, using the "smaller" groups described above, it is possible to form groups of students in each of which there are m people speaking English, m people speaking French and m people speaking German where m is any *even* number not exceeding n; here the condition that the number m is even is essential and cannot be discarded (try to prove this!). (By the way, it is also advisable to try to replace the number 3, the number of the languages in this problem, by some other number.)

25. (a) It is clear that the *least* possible value of the "average place" is equal to 1. This is the case when all the referees give the 1st place to *one and the same* athlete. On the other hand, 5 or more athletes cannot be given the first place by different referees. Indeed, if this were the case then these n ($n \geqslant 5$) athletes would be given altogether 9 first places by the nine referees and $9n - 9 \geqslant 9 \cdot 5 - 9 = 36$ other places (because the total number of places which the 9 referees give these n athletes is equal to $9n$). By the condition of the problem, none of these places can be worse than the 4th, which is impossible because the total number of places from the 2nd to the 4th given by the referees to the athletes is equal to $3\cdot 9 = 27$. Thus, it only remains to consider the cases when the first place is given (by different referees) to 2, 3 or 4 athletes.

1°. If the referees give the first place only to *two* athletes then at least one of them is given the first place by not less than five referees, and since the places given to that athlete by the other referees are not worse than the 4th, the "average place" of that athlete is not worse than $\frac{(5\cdot 1+4\cdot 4)}{9}=\frac{21}{9}=2\frac{1}{3}$.

2°. If *three* athletes are given the first places then these athletes are given altogether 9 first places and $3\cdot 9-9=18$ other places none of which can be worse than the fourth one. Since the 9 referees can only give the athletes 9 fourth places, in the "worst" case these athletes are given 9 fourth places and 9 third places. Thus, the sum of the values of the places given to the athletes is not more than $9\cdot 1+9\cdot 4+9\cdot 3=72$, and, therefore, the sum of the places of at least one of them does not exceed $72/3=24$ and his "average place" is not worse than $\frac{24}{9}=2\frac{2}{3}$.

3°. Finally, if the referees give the first place to *four* athletes then these four people are given altogether 9 first places and $4\cdot 9-9=27$ other places none of which can be worse than the 4th. Among these 27 places there can be 9 fourth places, 9 third places and 9 second places. Thus, the total sum of the places of these four athletes is equal to $9\cdot 1+9\cdot 2+9\cdot 3+9\cdot 4=90$ $(90<4\cdot 23)$; consequently the sum of the places of the best of these athletes does not exceed 22 and his "average place" is not worse than $\frac{22}{9}=2\frac{4}{9}<2\frac{2}{3}$.

Hence, the "average place" of the best athlete cannot be worse than $2\frac{2}{3}$; it can be equal to $2\frac{2}{3}$ only when each of the three best athletes is given the first place by three referees, the third place by some other three referees and the fourth place by the last three referees (in that case the three athletes become simultaneously the winners of the contest).

Remark. It is clear that if we replace the kth place in the above argument by the $(21-k)$th place, it will follow that the value of the "average place" of the worst athlete cannot be better than $18\frac{1}{3}$ $\left(\text{but it can be equal to } 18\frac{1}{3}\right)$.

(b) It is clear that after every round the number of the best of the remaining tennis-players does not decrease; it is also clear that this number can increase by not more than 2 (it can increase by 2 in case the best tennis-player accidentally loses a game to another tennis-player whose number exceeds by 2 that of the former). Since $1024 = 2^{10}$ and since the number of the participants of the contest decreases by half after every round the total number of the rounds is equal to 10. After the 10th round only $2^0 = 1$ tennis-player, the winner, remains. Since after every round the number of the best tennis-player can increase by 2 it may seem that the 21st tennis-player may win.

But the 21st tennis-player *cannot in fact be the winner.* Indeed, for the 21st tennis-player to win it is necessary that after every round the two best tennis-players should leave; in other words, in the first round the 1st and the 2nd tennis-players should lose to the 3rd and the 4th tennis-players respectively, in the second round the 3rd and the 4th tennis-players should lose to the 5th and the 6th tennis-players respectively and so on. This means that in the semi-finals the 17th and the 18th tennis-players should lose to the 19th and the 20th tennis-players respectively. Therefore the 19th and the 20th tennis-players take part in the finals and hence it is one of them that becomes the winner but not the 21st tennis-player.

Finally, let us show that the 20th tennis-player *can be the winner of the games.* Indeed, if there are only $2^1 = 2$ tennis-players then of course the winner can have the number $2 = 2 \cdot 1$; if there are $2^2 = 4$ tennis-players then the winner can have the number $2 \cdot 2 = 4$ because in the pair of the first two tennis-players the 2nd tennis-player can be the winner and in the pair of the other two tennis-players the 4th can be the winner, and, in principle, it is possible that in the finals the 4th tennis-player defeats the 2nd one. Similarly, if there are $2^3 = 8$ tennis-players then the 6th tennis-player can be the winner: indeed, if the four best tennis-players are in one sub-group then in this sub-group the 4th tennis-player can win while in the other sub-group the 6th tennis-player can defeat the 5th one; in the finals the 6th tennis-player can defeat the 4th one. Similarly, using the *method of mathematical induction* we can easily prove that if 2^n *tennis-players take part in the Olympic games then the* (2^n)*th tennis-player can become the winner*: for this to happen it is sufficient that the first $2n - 2$ tennis-players should be in one sub-group of 2^{n-1} tennis-players in which (by the induction hypothesis) the $(2n-2)$th tennis-player can win and that in the other sub-group the $(2n)$th tennis-player should win because, in principle, it is possible that in the finals the $(2n)$th tennis-player defeats the $(2n-2)$th tennis-player.

26. Let N_i denote the number of sets of medals remaining after the $(i-n)$th day, where $i = 1, 2, \ldots, n$. By the way, we can also assume that the quantity N_i makes sense for $i > n$: let it be equal to 0 in such cases. The conditions of the problem (and the above assumption) imply that $N_1 = N$, $N_n = n$ and $N_{n+i} = 0$; besides, N_i and N_{i+1} are connected by the relation

$$N_{i+1} = N_i - i - \frac{1}{7}(N_i - i) = \frac{6}{7}(N_i - i)$$

that is

$$N_i = \frac{7}{6} N_{i+1} + i \qquad (*)$$

Using (*) we find, in succession, that

$$N_n = n = \frac{7}{6} N_{n+1} + n$$

$$N_{n-1} = \frac{7}{6} n + (n-1)$$

$$N_{n-2} = \left(\frac{7}{6}\right)^2 n + \frac{7}{6}(n-1) + (n-2)$$

$$N_{n-3} = \left(\frac{7}{6}\right)^3 n + \left(\frac{7}{6}\right)^2 (n-1) + \frac{7}{6}(n-2) + (n-3)$$

. . . .

$$N_i = \left(\frac{7}{6}\right)^{(n-i)} n + \left(\frac{7}{6}\right)^{(n-i-1)} (n-1) + \left(\frac{7}{6}\right)^{(n-i-2)} (n-2) + \ldots + i$$

(the general formula can be of course readily proved with the aid of the method of mathematical induction).

Thus, we obtain:

$$N = N_1 = \left(\frac{7}{6}\right)^{n-1} n + \left(\frac{7}{6}\right)^{n-2} (n-1) + \left(\frac{7}{6}\right)^{n-3} (n-2) + \ldots$$

$$\ldots + 1 = n\left[\left(\frac{7}{6}\right)^{n-1} + \left(\frac{7}{6}\right)^{n-2} + \left(\frac{7}{6}\right)^{n-3} + \ldots + 1\right] -$$

$$-\left[\left(\frac{7}{6}\right)^{n-2} + 2\left(\frac{7}{6}\right)^{n-3} + \ldots + (n-2)\frac{7}{6} + (n-1)\right] = S_1 \cdot n - S_2$$

where S_1 and S_2 denote the sums in the square brackets. By the formula for the sum of the members of a geometric progression, we obviously have

$$S_1 = \left(\frac{7}{6}\right)^{n-1} + \left(\frac{7}{6}\right)^{n-2} + \ldots + 1 = \frac{\left(\frac{7}{6}\right)^n - 1}{\frac{7}{6} - 1} = 6\left[\left(\frac{7}{6}\right)^n - 1\right]$$

On the other hand, it can readily be seen that

$$\left(\frac{7}{6}\right) \cdot S_2 - S_2 = \left(\frac{7}{6}\right)^{n-1} + \left(\frac{7}{6}\right)^{n-2} + \ldots + \left(\frac{7}{6}\right) - (n-1) =$$

$$= S_1 - n = 6\left[\left(\frac{7}{6}\right)^n - 1\right] - n$$

whence

$$S_2 = 36\left[\left(\frac{7}{6}\right)^n - 1\right] - 6n$$

and, consequently,

$$N = 6\left[\left(\frac{7}{6}\right)^n - 1\right]n - 36\left[\left(\frac{7}{6}\right)^n - 1\right] + 6n = 6(n-6)\left(\frac{7}{6}\right)^n + 36 \quad (^{**})$$

Since N is an integral number, the number $7^n(n-6)/6^{n-1}$ must be integral and therefore so must be the number $(n-6)/(6^n-1)$. Therefore n is a multiple of 6. On the other hand, it is evident that for all $k \geqslant 2$ the inequality $6k-6 < 6^{6k-1}$ holds (why?), that is $k-1 < 6^{6k-2}$. It follows that the expression $(n-6)/6^{n-1}$ cannot be an integral number for $n > 6$. We thus arrive at the single solution of the problem: $n = 6$, and therefore, by (**), $N = 36$.

27. *First solution.* Let n denote the number of nuts each of the friends got in the morning. Then the number of the nuts in the bag the friends found in the morning was equal to $5n+1$. The last of the friends who woke up at night obviously took $\frac{5n+1}{4}$ nuts and before that there were $5\cdot\frac{5n+1}{4}+1 = \frac{25n+9}{4}$ nuts in the bag. The one but last of the friends who woke up at night took $\frac{1}{4}\cdot\frac{25n+9}{4}$ nuts and before that there were $5\cdot\frac{1}{4}\cdot\frac{25n+9}{4}+1 = \frac{125n+61}{16}$ nuts in the bag. The third friend took $\frac{1}{4}\cdot\frac{125n+61}{16}$ nuts and before that there were $5\cdot\frac{1}{4}\cdot\frac{125n+61}{16}+1=\frac{625n+369}{64}$ nuts in the bag. The second friend took $\frac{1}{4}\cdot\frac{625n+369}{64}$ nuts and before that there were $5\cdot\frac{1}{4}\cdot\frac{625n+369}{64}+1=\frac{3125n+2101}{256}$ nuts. Finally, the first friend took $\frac{1}{4}\cdot\frac{3125n+2101}{256}$ nuts, and the original number of the nuts in the bag was equal to

$$N = 5\cdot\frac{1}{4}\cdot\frac{3125n+2101}{256}+1=\frac{15\,625n+11\,529}{1024} = 15n+11+\frac{265n+265}{1024}$$

Since the number N must be integral the number $265(n+1)$ must be divisible by 1024. The smallest number n satisfying this condition is obviously equal to 1023, and in that case

$$N = 15\cdot 1023 + 11 + 265 = 15\,621$$

Second solution. This problem can be solved in a simpler way and almost without calculations if we consider the requirements which are imposed on the total number N of the nuts by the conditions of the problem. The first condition of the problem is that when the nuts in the bag are first divided into five parts there remains one nut; hence, the division of N by 5 must leave a remainder of 1, that is $N = 5l + 1$. The difference between any two neighbouring numbers of the form $5l + 1$ is equal to $+5$ or to -5, and knowing one of the numbers we can find infinitely many other numbers of this form by adding to that number (or subtracting from it) any numbers multiple of 5. The second condition of the problem implies that when number $k = \frac{4}{5}(N-1) = 4l$ is divided by 5 the remainder is equal to 1, that is $k = 5l_1 + 1$. This condition is equivalent to the requirement that when l is divided by 5 the remainder should be equal to 4 whence it follows that when $N = 5l + 1$ is divided by 25 the remainder should be equal to 21. The difference between any two neighbouring numbers satisfying this requirement is equal to $+25$ or to -25, and knowing one such number we can obtain an arbitrary set of these numbers by adding to that number (or by subtracting from it) any number multiple of 25. Similarly, the third condition of the problem implies that when the number $k_1 = \frac{4}{5}(k-1) = 4l_1$ is divided by 5 the remainder is equal to 1. This condition determines the remainder resulting from the division of l_1 by 5, the remainders resulting from the division of k and l by 25 and the remainder resulting from the division of N by 125. All the conditions of the problem determine the remainder resulting from the division of N by $5^6 = 15\,625$. The difference between any two neighbouring numbers satisfying these conditions is equal to $+15\,625$ or to $-15\,625$.

It is possible to calculate the remainder resulting from the division of the number N by 5^6 but we can do without it. The matter is that one of the numbers satisfying all the conditions of the problem can readily be indicated: it is the number -4.

Indeed, when -4 is divided by 5 we obtain -1 in the quotient and $+1$ in the remainder. Therefore if we subtract the number 1 from -4 and then take 4/5 of the resultant difference which is divisible by 5 we again obtain the same number -4. Similarly, all the following divisions by 5 will leave the same remainder $+1$. However, the number -4 itself cannot be the answer to the problem because N must be a positive number. But knowing one of the numbers satisfying the conditions of the problem we can find an arbitrary number of others by adding to that number any numbers multiple of 5^6. The smallest positive integer satisfying the conditions is obviously equal to $-4 + 5^6 = 15\,625 - 4 = 15\,621$.

28. Let n denote the number of the sheep in the flock. Then the brothers got n rubles for every sheep and, consequently, the total cash they got was $N = n \cdot n = n^2$ rubles. Let d be the quotient resulting from the division of n by 10 and let e be the digit in the ones place of the number n; then $n = 10d + e$ and

$$N = (10d + e)^2 = 100d^2 + 20\,de + e^2$$

The conditions of the problem imply that the quotient resulting from the division by 10 of the number of rubles the elder brother got exceeds by unity the quotient resulting from the division by 10 of the number of rubles the younger brother got, whence it follows that the quotient resulting from the division by 10 of the number N is *odd*. The number $100d^2 + 20de = 20d(5d + e)$ is divisible by 20 and therefore the quotient resulting from the division of this number by 10 is even. Therefore, when the number e^2 is divided by 10 we must obtain an odd number in the quotient. Since e is less than 10, the number e^2 can only assume one of the values

$$1;\ 4;\ 9;\ 16;\ 25;\ 36;\ 49;\ 64\ \text{ and }\ 81$$

Among these numbers only 16 and 36 have odd digits in their tens places. Consequently, e^2 is equal either to 16 or to 36. Both numbers end with 6, and hence when the younger brother's last turn to take his money came he got 6 rubles instead of 10 and thus the elder brother got 4 rubles more than the younger brother. Therefore, for the sharing to be fair the elder brother must pay 2 rubles to his younger brother, whence it follows that the knife cost 2 rubles.

29. (a) In the Gregorian calendar (which is now in general use) every year has 365 days with the exception of the leap years. Each leap year has an additional (the 366th) day (the 29th of February). The leap years are those divisible by 4 except the years divisible by 100 but not divisible by 400. These years (for instance, 1800, 1900 and 2100) have 365 (but not 366) days each and they are not leap years; for instance, the year 2000 is a leap year since the number 2000 is divisible by 400. A New Year's Day is on the 1st of January, and so we have to find which of the two days, Saturday or Sunday, happens to be more frequently the 1st of January.

The intervals between the days which are the 1st of January are not always constant but they vary periodically with period of 400 years. Four hundred years consist of an integral number of weeks. Indeed, the common year consists of 52 weeks plus one more day and the leap year consists of 52 weeks plus two days. A period of four years one of which is a leap year consists of $4 \cdot 52$ weaks plus 5 more days. Since a period of 400 years includes three years which are divisible by 100 and are not divisible by 400 such a period consists of $400 \cdot 52$ weeks plus $5 \cdot 100 - 3 = 497$ days$=71$

weeks, that is of an integral number of weeks. Therefore it is sufficient to find which of the two days, Saturday or Sunday, happens te be more frequently the 1st af January during any period of 400 years; for any other such period the answer to the question is the same.

Let us consider the period of 400 years from 1901 to 2301. It should be noted that if during a period of 28 years every fourth year is a leap year, that is if these 28 years do not contain a year which is divisible by 100 and not divisible by 400, then these 28 years consist of an integral number of weeks because each subinterval of four years consists of an integral number of weeks plus 5 days and the whole period of 28 years consists of an integral number of weeks plus $5 \cdot 7 = 35$ days $= 5$ weeks. Now we note that the 1st of January of 1952 was Tuesday. Since each common year consists of an integral number of weeks plus one day and the leap year consists of an integral number of weeks plus 2 days, the 1st of January of 1953 was Thursday (because 1952 was a leap year), the 1st of January of 1954 was Friday, the 1st of January of 1955 was Saturday and so on. We similarly find that the 1st of January of 1951 was Monday, the 1st of January of 1950 was Sunday and so on. In this way we find that during the 28 years from 1929 to 1956 the 1st of January happened to be equally frequently (exactly 4 times) each of the seven days of the week. Exactly the same distribution of the days of the week which were New Year's Days was during the 28 years from 1901 to 1928 (we remind the reader that if during a period of 28 years every fourth year is a leap year then this period consists of an integral number of weeks, and therefore the distribution of the days of the week which are New Year's Days during such periods of 28 years is one and the same). The same distribution of the days of the week which were New Year's Days must have been during the periods from 1957 to 1984, from 1985 to 2012 (because the year 2000 will be a leap year since the number 2000 is divisible by 400), from 2013 to 2040, from 2041 to 2068 and from 2069 to 2096. Thus, during the period from 1901 to 2096 the 1st of January happens to be equally frequently every day of the week.

Further, the 1st of January of 2097 will be the same day of the week as the 1st of January of 1901 or the 1st of January of 1929, that is Tuesday. The 1st of January of 2098 will be Wednesday, the 1st of January of 2099 will be Thursday, the 1st of January of 2100 will be Friday and the 1st of January of 2101 will be Saturday (because the year 2100 will not be a leap year). The next period of 28 years will differ from the period from 1901 to 1928; that period will start on Saturday instead of Tuesday; however, since during the 28 years from 1901 to 1928 the 1st of January was exactly 4 times every day of the week, during the period from 2101 to 2128 the 1st of January will again be exactly 4 times every day of the week. What has been said also refers to the periods from 2129 to 2156 and from 2157 to 2184; the year 2185 will begin with the same day as 2101, that is with Saturday. This makes it possible to find what days of the week will be the 1st of January of 2185 and 2201. Simple calculations show that during the period from 2185 to 2200 the 1st of January will be exactly twice Monday, Wednesday, Thursday, Friday and Saturday and exactly 3 times Sunday and Tuesday. The 1st of January of 2201 will be Thursday. During $3 \cdot 28 = 84$ years from 2201 to 2284 the 1st of January will happen to be equally frequently every day of the week. The 1st of January of 2285 will be the same day as the 1st of January of 2201, that is Thursday. This makes it possible to describe the distribution of the days of the week with which the years will begin during the period from 2285 to 2300. It turns out that during this period the 1st of January will be exactly twice Monday, Tuesday, Wednesday, Thursday and Saturday and exactly three times Sunday and Friday. Thus, in addition to the periods during which the 1st of January happens to be equally frequently every day of the week we have $2 + 2 = 4$ Mondays, $1 + 3 + 2 = 6$ Tuesdays, $1 + 2 + 2 = 5$ Wednesdays, $1 + 2 + 2 = 5$ Thursdays, $1 + 2 + 3 = 6$ Fridays, $2 + 2 = 4$ Saturdays and $3 + 3 = 6$ Sundays. It follows that the 1st of January is more frequently Sunday than Saturday.

(b) By analogy with the solution of Problem 29 (a), we can show that during any period of 400 years the 30th day of a month happens to be Sunday 687 times, Monday 685 times, Tuesday 685 times, Wednesday 687 times, Thursday 684 times, Friday 688 times and Saturday 684 times. Thus, most frequently the 30th day of a month happens to be Friday.

30. It can readily be seen that when the last digit of a number is deleted the number decreases not less than 10 times. A number which decreases exactly 10 times when its last digit is deleted must have nought at the end; consequently all such numbers satisfy the condition of the problem.

Now let us suppose that a whole number x decreases more than 10 times when its last digit is deleted, namely let it decrease $10 + a$ times $(a \geqslant 1)$. Let y be the quotient resulting from the division of the number x by 10 and let z be the digit in the ones place of the number $x: x = 10y + z$. After the last digit of the number x is deleted we obtain the number y; therefore the conditions of the problem imply

$$x = (10 + a) \cdot y$$

that is

$$10y + z = (10 + a) \cdot y$$

whence

$$z = ay$$

Since $z < 10$ we have $y < 10$ and $a < 10$. Consequently, the numbers possessing the required property have only two digits and can decrease not more than 19 times when the last digit is deleted. Now it can easily be seen that a number which decreases 11 times when its last digit is deleted can be equal only to 11; 22; 33; 44; 55; 66; 77; 88 and 99. Indeed, if $10 + a = 11$ then $a = 1$; consequently, $z = ay = y$ and $x = 10y + z = 11y$ where $y = 1, 2, 3, \ldots, 9$. We similarly find that the numbers decreasing 12 times can only be equal to 12, 24, 36 and 48 and the numbers decreasing 13 times are 13, 26 and 39. Analogously, the numbers decreasing 14 times are 14 and 28, and the numbers decreasing 15, 16, 17, 18 and 19 times can only be equal to 15; 16; 17; 18 and 19 respectively.

31. (a) Let the sought-for number have $k+1$ digits; then it can be written in the form $6 \cdot 10^k + y$ where y is a k-digit number (which may begin with one or several noughts). By the condition of the problem, we have

$$6 \cdot 10^k + y = 25 \cdot y$$

whence

$$y = \frac{6 \cdot 10^k}{24}$$

It follows that k cannot be less than 2 (if otherwise, $6 \cdot 10^k$ would not be divisible by 24). For $k \geqslant 2$ the number y is equal to $25 \cdot 10^{k-2}$, that is it has the form $25\underbrace{0\ldots0}_{(k-2)\text{ noughts}}$, Therefore all the sought-for numbers are of the form $625\underbrace{0\ldots0}_{n\text{ noughts}}$ $(n = 0, 1, 2, \ldots)$.

(b) Let us apply the method used in Problem 31 (a) to the problem of finding a number which begins with a given digit a and decreases 35 times when this digit is deleted. Then we arrive at the equality

$$y = \frac{a \cdot 10^k}{34}$$

where y is a whole number (see the solution of Problem 31 (a)). It now becomes obvious that there are no numbers $a \leqslant 9$ and k for which this equality holds.

Remark. By complete analogy with the solutions of Problems 31 (a) and (b), we can show that a number beginning with a known digit a decreases an integral number of times b when this digit a is deleted only in the case when *$b-1$ is a number exceeding a such that all the prime factors of the number $b-1$ different from 2 and 5 are also contained in the number a, the exponents of the powers of these factors being not less than those of the number $b-1$* (that is $a/(b-1)$ is a proper fraction which can be changed to a terminating decimal). For instance, there is no number which decreases 85 times when its first digit is deleted (because $85-1=84$ is divisible by $3\cdot 7$ whereas there is no digit that can be simultaneously divisible by 3 and 7), and a number which decreases 15 times when its first digit is deleted must begin with the digit 7 ($15-1=14$ is divisible by 7). We can readily derive the general expression for the numbers beginning with a known initial digit a which decrease a given number of times b when that initial digit is deleted.

32. (a) First of all let us show that if a number N decreases 9 times when one of its digits is deleted then this digit must be the first or the second one. Indeed, if otherwise, then, on writing

$$a_0\cdot 10^n+a_1\cdot 10^{n-1}+\ \dots\ +a_n=N$$

where $a_0, a_1, \dots, a_n$ are the digits of the number N, we conclude that $N/9$ has n digits the first two of which are a_0 and a_1, that is

$$a_0\cdot 10^{n-1}+a_1\cdot 10^{n-2}+\ \dots\ =\frac{N}{9}$$

The multiplication of the last equality by 10 and the subtraction of the first equality from that product obtained yield

$$\frac{N}{9}<10^{n-1}$$

The last inequality cannot hold because

$$\frac{N}{10}=a_0\cdot 10^{n-1}+\ \dots\ \geqslant 10^{n-1}$$

On the other hand, the test for the divisibility by 9 implies that if a number N and the number obtained from N by deleting one of its digits are simultaneously divisible by 9 then this digit is either 0 or 9. Thus, in the case under consideration the first or the second digit of the number N can only be equal to 0 or 9, and the deletion of this digit is equivalent to the division of N by 9. However, the initial digit of the number N cannot be equal to 0, and if it were equal to 9 the number $N/9$ would have the same number of digits as N and could not be obtained from N by deleting one digit. Further, if the second digit of the number N is equal to 9 and if the number obtained from N by deleting this digit is equal to $N/9$ then we have

$$N=a_0\cdot 10^n+9\cdot 10^{n-1}+a_2\cdot 10^{n-2}+\ \dots\ +a_n$$

and

$$\frac{N}{9}=a_0\cdot 10^{n-1}+a_2\cdot 10^{n-2}+\ \dots\ +a_n$$

Now, on multiplying the second of these equalities by 10 and subtracting the first equality from the resulting product, we obtain

$$\frac{N}{9}<10^{n-1}$$

(because $a_2 \leqslant 9$). Thus, we see that *in order to decrease the number N nine times we must delete the second digit which is equal to* 0.

Now we have

$$N = a_0 \cdot 10^n + a_2 \cdot 10^{n-2} + \ldots + a_n$$

and

$$\frac{N}{9} = a_0 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \ldots + a_n$$

It follows that

$$\frac{N}{9} = N - a_0 \cdot 10^n + a_0 \cdot 10^{n-1} = N - a_0 \cdot 10^{n-1} \cdot 9$$

and, finally,

$$\frac{1}{9} \cdot \frac{N}{9} = \frac{N}{9} - a_0 \cdot 10^{n-1}$$

which means that in order to divide the number $N/9$ by 9 it is sufficient to delete its initial digit.

(b) We have (see the solution of Problem 32 (a))

$$\frac{N}{9} = N - a_0 \cdot 10^{n-1} \cdot 9$$

whence it readily follows that

$$N = \frac{a_0 \cdot 10^{n-1} \cdot 81}{8}$$

Now, making a_0 assume, in succession, the values 1, 2, 3, etc. we conclude that the number N can be equal to one of the numbers 10 125; 2025; 30 375; 405; 50 625; 6075 and 70 875 or it can differ from one of these numbers in a group of noughts placed at the end (a_0 cannot be equal to 8 or 9 because in this case the second digit of N cannot be equal to 0).

33. (a) Let us suppose that a whole number N decreases m times when its third digit is deleted. Then, by analogy with the solution of Problem 32 (a), we can write

$$N = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \ldots + a_n$$

and

$$10 \cdot \frac{N}{m} = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \ldots + a_n \cdot 10$$

For $m < 10$ we obtain $(10 - m)N/m < 10^{n-1}$, which is impossible because $(10 - m)/m > 1/10$ and $N/10 = a_0 \cdot 10^{n-1} + \ldots \geqslant 10^{n-1}$. For $m > 11$ we obtain $(m - 10)N/m < 10^{n-1}$, which is impossible (the same reason: $(m - 10)/m > 1/10$). Finally, if $m = 11$, there must be $N/11 < 10^{n-1}$, that is the number of digits of

$N/m = N/11$ is less by two than the number of digits of N, which is impossible.

Hence, the only possible case is $m = 10$; consequently, the condition of the problem is satisfied by those and only those numbers whose all digits except the first two are noughts.

Remark. We can analogously show that the whole numbers which decrease an integral number of times when their kth digits are deleted (where $k > 3$) are those whose all digits except the first $k - 1$ digits are noughts.

(b) By analogy with the solution of Problem 32, we can write the equalities

$$N = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \dots + a_n$$

and

$$\frac{N}{m} = a_0 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \dots + a_n$$

for a whole number N which decreases m times when its second digit is deleted. It follows that

$$\frac{N}{m} = N - a_0 \cdot 10^n - a_1 \cdot 10^{n-1} + a_0 \cdot 10^{n-1}$$

whence, after simple transformations, we obtain

$$N = \frac{(9a_0 + a_1) \cdot 10^{n-1} \cdot m}{m - 1} \qquad (*)$$

The last relation can be rewritten in the form

$$N = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} - a_0 \cdot 10^{n-1} + \frac{9(a_0 + a_1) \cdot 10^{n-1}}{m-1}$$

On the other hand, we know that N is an $(n + 1)$-digit number whose first two digits are a_0 and a_1, that is

$$N = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \dots + a_n$$

where we can assume that not all the digits $a_2, \dots, a_n$ are noughts (if otherwise, the problem reduces to the consideration of two-digit numbers N; see the solution of Problem 30). We see that the inequalities

$$0 < -a_0 \cdot 10^{n-1} + \frac{(9a_0 + a_1) \cdot 10^{n-1}}{m-1} < 10^{n-1}$$

must hold; they are equivalent to the inequalities

$$a_0 < \frac{9a_0 + a_1}{m-1} < a_0 + 1 \qquad (**)$$

Thus, we can finally state the following results. The sought-for numbers N are expressed by formula (*) where $0 \leqslant a_0 \leqslant 9$ and $0 \leqslant a_1 \leqslant 9$; since N is a whole number and the numbers m and

$m-1$ are relatively prime, it follows that the proper fraction $(9a_0+a_1)/(m-1)$ can be changed to a terminating decimal; the admissible values of a_0, a_1 and m must satisfy inequalities (**). Besides, to these possible values of N we must add the two-digit numbers obtained in the solution of Problem 30.

Now it only remains to consider consecutively all the possible values of a_0.

1°. $a_0 = 1$. In this case inequalities (**) result in

$$1 < \frac{18}{m-1}, \quad m-1 < 18; \quad \frac{9}{m-1} < 2 \text{ and } m-1 > 4$$

On making $m-1$ assume, in succession, the values 5, 6, 7, ..., 17 and choosing every time the appropriate values of a_1 we obtain the following values of N:

$$N = 108;\ 105;\ 10\,125;\ 1125;\ 12\,375;\ 135;\ 14\,625;\ 1575;$$
$$16\,875;\ 121;\ 132;\ 143;\ 154;\ 165;\ 176;\ 187;\ 198;$$
$$1625;\ 195;\ 192;\ 180\,625;\ 19\,125$$

To each of these numbers we can add an arbitrary number of noughts at the end.

Further, in a similar manner we obtain:

2°. $a_0 = 2$:

$$N = 2025;\ 21\,375;\ 225;\ 23\,625;\ 2475;\ 25\,875;\ 231;\ 242;$$
$$253;\ 264;\ 275;\ 286;\ 297;\ 2925$$

3°. $a_0 = 3$:

$$N = 30\,725;\ 315;\ 32\,625;\ 3375;\ 34\,875;\ 341;\ 352;\ 363;$$
$$374;\ 385;\ 396$$

4°. $a_0 = 4$:

$$N = 405;\ 41\,625;\ 4275;\ 43\,875;\ 451;\ 462;\ 473;\ 484;\ 495$$

5°. $a_0 = 5$:

$$N = 50\,625;\ 5175;\ 52\,875;\ 561;\ 572;\ 583;\ 594$$

6°. $a_0 = 6$:

$$N = 6075;\ 61\,875;\ 671;\ 682;\ 693$$

7°. $a_0 = 7$:

$$N = 781;\ 792$$

8°. $a_0 = 8$:

$$N = 891$$

There are no values of N which correspond to $a_0 = 9$.

Altogether, including the results of Problem 30, we obtain for the number N one hundred and four values to each of which we can add an arbitrary number of noughts at the end.

34. (a) *First solution.* Let us denote as X the (m-digit) number which is obtained when the initial digit 1 is deleted in the sought-for number. Then, by the condition of the problem, we have

$$(1 \cdot 10^m + X) \cdot 3 = 10X + 1$$

whence

$$X = \frac{3 \cdot 10^m - 1}{7}$$

From the last equality we can easily find the number X. To this end let us consider the process of long division of the number $3 \cdot 10^m = 30\,000\ldots$ by 7 until 1 is obtained in the remainder. We have:

$$\begin{array}{r l}
 & 42\,857 \\
7) & \overline{3000\ldots0} \\
 & -28 \\
 & \overline{20} \\
 & -14 \\
 & \overline{60} \\
 & -56 \\
 & \overline{40} \\
 & -35 \\
 & \overline{50} \\
 & -49 \\
 & \overline{1}
\end{array}$$

Thus, the least possible value of the number X is equal to 42 857 and the least possible value of the sought-for number is 142 857.

After the first digit 1 is obtained, the process of long division could be continued until the next digit 1 is obtained and so on. This would result in the numbers of the form

$$\underbrace{\underbrace{142\,857}\;\underbrace{142\,857}\ldots\underbrace{142\,857}}_{k \text{ times}}$$

which also satisfy the condition of the problem.

Second solution. Let us denote the second digit of the sought-for number as x, the third digit as y etc., that is let us suppose that the sought-for number has the form $\overline{1xy\ldots zt}$ (the bar above this expression means that we deal here with a number whose digits are $1, x, y, \ldots, z, t$ but not with the product $1 \cdot x \cdot y \ldots z \cdot t$). Then, by the condition of the problem, we have

$$\overline{1xy\ldots zt} \cdot 3 = \overline{xy\ldots zt1}$$

It follows that $t = 7$ (if otherwise, the product on the left-hand side could not end with 1). Consequently, the digit in the tens place of the number on the right-hand side is equal to 7. This is only possible if the product $z \cdot 3$ ends with $7 - 2 = 5$ (here the number 2 which is subtracted from 7 appears due to the product of the last digit 7 of the sought-for number by 3), that is $z = 5$. We have thus found that the digit in the hundreds place in the number on the right-hand side is equal to 5; therefore the multiplication of the digit in the hundreds place of the sought-for number by 3 must result in a number which ends with $5 - 1 = 4$ (here 1 is the digit in the tens place of the product $5 \cdot 3$). These calculations finish when we arrive at the first digit 1. The calculation process can be conveniently represented by arranging the operations in the following way:

$$\begin{array}{cccccccc} 1 & 4 & 2 & 8 & 5 & 7 & & 42\,857 \\ . & . & . & . & . & . \times 3 = & . \; . \; . \; . \; . & 1 \\ 4-\dot{1}=3; & 2-\dot{0}=2; & 8-\dot{2}=6; & 5-\dot{1}=4; & 7-2=5 & & & \end{array}$$

(the calculations are carried *from right to left*). Thus, the least number satisfying the conditions of the problem is 142 857.

If these calculations are continued after the first unity is obtained we find the other numbers satisfying the conditions of the problem:

$$\underbrace{\underbrace{142\,857}\ \underbrace{142\,857}\ \dots\ \underbrace{142\,857}}_{k \text{ times}}$$

(b) Since the number of the digits does not increase when the whole number in question is increased three times, it follows that the initial digit of that number can only be equal to 1, 2, or 3.

As is seen from the solution of Problem 34 (a), it is possible that this digit is equal to 1. Now let us show that it cannot be equal to 3.

Indeed, if the initial digit of the sought-for number were equal to 3 then its second digit (which coincides with the first digit of the number equal to the given number times three) would be equal to 9. But the number obtained when a number beginning with the digits 39 is multiplied by 3 has more digits than the original number itself; therefore it cannot be obtained from the original number by carrying its initial digit to the end.

Let the reader prove that the sought-for numbers *can* begin with the digit 2. The *smallest* of these numbers is 285 714; all such numbers beginning with the digit 2 have the form

$$\underbrace{\underbrace{285\,714}\ \underbrace{285\,714}\ \dots\ \underbrace{285\,714}}_{k \text{ times}}$$

(the proof is analogous to the solution of Problem 34 (a)).

35. *First solution.* Let X be the number satisfying the conditions of the problem. Then we have

$$X=\overline{a_1a_2\ldots a_{n-1}6} \quad\text{and}\quad 4X=\overline{6a_1a_2\ldots a_{n-1}}$$

where $a_1, a_2, \ldots, a_{n-1}$ and 6 are the digits of the number X. Since the last digit of the number X is 6, the last digit a_{n-1} of the number $4X$ is 4; hence, $X=\overline{a_1a_2\ldots a_{n-2}46}$; this makes it possible to find the last but one digit $a_{n-2}=8$ of the number $4X$. Now, on writing X in the form $\overline{\ldots 846}$ we can determine $a_{n-3}=3$ etc. Let us continue this process until the digit 6 is obtained in the number $4X$; this digit can be regarded as being carried from the end of the number X. In this way we find that the *smallest* number satisfying the condition of the problem is $X=153\,846$ and that $4X=615\,384$.

Second solution. Since the sought-for number X has the digit 6 at its end, it can be written in the form $X=10x+6$ where x is the number obtained from X by deleting that last digit 6. If x is an n-digit number, the conditions of the problem imply that

$$4\cdot(10x+6)=6\cdot 10^n+x$$

that is

$$39x=6\cdot(10^n-4) \quad\text{whence}\quad x=\frac{2(10^n-4)}{13} \qquad (*)$$

The number 10^n-4 is obviously equal to 6 or to 96, or to 996, or to 9996, ... The smallest of these numbers which is multiple of 13 is the number $99\,996=13\cdot 7692$, and the value of n corresponding to it is equal to 5. It follows that equality (*) results in $x=15\,384$ and, consequently, $X=153\,846$.

36. If the multiplication of a number by 5 does not change the number of its digits, the initial digit of the number must be 1. When this digit is carried to the end we obtain a number whose last digit is 1. But such a number cannot be divisible by 5.

In a similar way it can be proved that there are no numbers which increase 6 or 8 times when their initial digits are carried to the end.

37. *First solution.* Since the product of the sought-for number by 2 has the same number of digits as the original number, the initial digit of that number cannot exceed 4. When the initial digit is carried to the end the resultant number must be even (it is equal to the duplicated original number), and therefore the initial digit of the sought-for number must be even. Hence, it can only be equal to 2 or 4.

Now let us suppose that the initial digit of the sought-for number is equal to 2 or 4. On denoting as X the number obtained from the sought-for number by discarding its initial digit, we can write, by analogy with the first solution of Problem 34 (a), the equality

$$(2\cdot 10^m + X)\cdot 2 = 10\cdot X + 2 \quad \text{whence} \quad X = \frac{4\cdot 10^m - 2}{8} = \frac{2\cdot 10^m - 1}{4}$$

or

$$(4\cdot 10^m + X)\cdot 2 = 10\cdot X + 4 \quad \text{whence} \quad X = \frac{8\cdot 10^m - 4}{8} = \frac{2\cdot 10^m - 1}{2}.$$

Now we see that neither of the formulas for the number X we have derived can hold because a whole number cannot be equal to a fraction whose numerator is odd and denominator is even.

Second solution. As in the first solution, we conclude that the initial digit of the sought-for number can only be equal to 2 or 4. Further, using the notation introduced earlier (see the second solution of Problem 34 (a)) we can write

$$\overline{2xy\ldots zt}\cdot 2 = \overline{xy\ldots zt2} \quad \text{or} \quad \overline{4xy\ldots zt}\cdot 2 = \overline{xy\ldots zt4}$$

From the first of these relations it follows that t can only be equal to 1 or 6 (because, if otherwise, the product on the left-hand side could not end with 2). However, if $t = 1$ then on the left-hand side we obtain a number which is not divisible by 4 whereas on the right-hand side a number divisible by 4 (because its last two digits are 12). If $t = 6$ then, on the contrary, we obtain a number divisible by 4 on the left-hand side and a number which is not divisible by 4 (because its last two digits are 62) on the right-hand side.

From the second of the last two relations it follows that t can be equal to 2 or 7. If $t = 2$ then, by analogy with the second solution of Problem 34 (a), we find that $z = 1$ or $z = 6$; for $z = 1$ the product on the left-hand side is divisible by 8 (since it is equal to the product of a number whose last two digits are 12 by the number 2) whereas the number on the right-hand side is not divisible by 8 (because it ends with 124). For $z = 6$ the number on the right-hand side is divisible by 8 whereas the product on the left-hand side is divisible by 4 but not by 8. It can similarly be shown that t cannot be equal to 7.

38. (a) *First solution.* A number which increases 7 times when its initial digit is carried to the end must begin with the digit 1 (if otherwise, the number which is 7 times as great as the original number must have more digits than the original number). Further, on denoting by X the m-digit number obtained from the original number by discarding its initial digit we can write (cf. the solution of Problem 34 (a)) the equality

$$(1\cdot 10^m + X)\cdot 7 = 10\cdot X + 1$$

whence

$$X = \frac{7\cdot 10^m - 1}{3}$$

Now it becomes clear that for any m X cannot be an m-digit number because $(7\cdot 10^m - 1)/3 > 10^m$.

We can similarly prove that there are no numbers which increase 9 times when their initial digits are carried to the end.

Second solution. As in the first solution, we conclude that the sought-for number can only have 1 as its initial digit. Further, using the notation introduced earlier we can write

$$\overline{1xy\ldots zt}\cdot 7=\overline{xy\ldots zt1}$$

It follows immediately that the last digit of the product $t\cdot 7$ is 1. Consequently, $t = 3$. On substituting this value of t into the above equality we obtain $\overline{1xy\ldots z3}\cdot 7 = \overline{xy\ldots z31}$. Since we have $3\cdot 7 = 21$ and since the product of the number $\overline{z3}$ by 7 ends in 31, the product $z\cdot 7$ must have 1 as its last digit. Consequently, z is equal to 3. In just the same way we can prove that every consecutive digit of the number in question is equal to 3 (it is meant here that the digits are read from right to left). At the same time, the initial digit must be equal to 1, which can never be achieved. Therefore there are no numbers which increase 7 times when their initial digits are carried to the end.

It can similarly be shown that there are no numbers increasing 9 times when their initial digits are carried to the end.

(b) *First solution.* Since the product of the sought-for number by 4 has not more digits than the original number, the initial digit of the original number cannot be greater than 2. When the initial digit is carried to the end we obtain an even number and therefore that initial digit must be equal to 2. Further, on denoting by X the m-digit number obtained when the initial digit of the sought-for number is discarded, we obtain

$$(2\cdot 10^m + X)\cdot 4 = 10X + 2 \quad \text{whence} \quad X = \frac{8\cdot 10^m - 2}{6}$$

This relation is impossible because $(8\cdot 10^m - 2)/6 > 10^m$ (cf. the solution of Problem 38 (a)).

Second solution. As in the first solution, we conclude that the initial digit of the sought-for number can only be equal to 2. Further, we have

$$\overline{2xy\ldots zt}\cdot 4=\overline{xy\ldots zt2}$$

whence it follows that $t = 3$ or $t = 8$ since $t\cdot 4$ ends in 2.

If t were equal to 8, the number on the right-hand side would end with 82 and therefore it would not be divisible by 4. In case $t = 3$ we have

$$\overline{2xy\ldots z3}\cdot 4=\overline{xy\ldots z32}$$

whence

$$\overline{2xy\ldots z0}\cdot 4=\overline{xy\ldots z20}$$

and

$$\overline{2xy\ldots z}\cdot 4=\overline{xy\ldots z2}$$

Thus, we see that the number $\overline{2xy\ldots z}$ possesses the same property as $\overline{2xy\ldots zt}$. Therefore, using the same argument, we conclude that $z = 3$. On continuing these calculations from right to left we consecutively find the digits and see that the decimal representation of the number in question involves only the digits 3. On the other hand, this number must have 2 as its initial digit, and consequently such a number does not exist.

39. *First solution.* Let us denote by $x, y, \ldots, z, t$ the unknown digits of the sought-for number. Using the notation of the second solution of Problem 34 (a) we can write

$$\overline{7xy\ldots zt}\cdot\frac{1}{3}=\overline{xy\ldots zt7}$$

whence

$$\overline{xy\ldots zt7}\cdot 3=\overline{7xy\ldots zt}$$

Now it becomes clear that $t = 1$; after that we can determine the digit z ($17\cdot 3$ ends in 51, and therefore $z = 5$). In this way, moving from right to left, we can consecutively find the digits of the sought-for number. The calculations should be stopped when we arrive at the digit 7. It is convenient to arrange the calculations in the following way:

$$\begin{array}{ccc}2413793103448275862068965517 & & 7241379310344827586206896551\\ \ldots\ldots\ldots\ldots\ldots7\cdot 3 = & & \ldots\ldots\ldots\ldots\ldots\end{array}$$

(the calculations are carried out *from right to left*). Thus, the least number satisfying the conditions of the problem is 7 241 379 310 344 827 586 206 896 551.

If the calculation process is continued after the first digit 7 is obtained we find the other numbers satisfying the condition of the problem. All such numbers have the form

$$\underbrace{\underbrace{7241379310344827586206896551}\ldots\underbrace{7241379310344827586206896551}}_{k\ \text{times}}$$

Second solution. Let $\overline{7xyz\ldots t}$ be the sought-for number. Then its division by 3 results in the number $\overline{xyz\ldots t7}$. Let us write the division process in the form

$$\begin{array}{r}\overline{xyz\ldots t7}\\ 3\overline{)\ 7xyz\ldots t}\end{array}$$

It follows that $x = 2$. If we substitute 2 for x into the dividend and into the quotient this will allow us to determine the second digit of the quotient; using this digit we can then find the third digit of the dividend; this makes it possible to determine the third digit of the quotient etc. The process ends when the last digit we obtain in the quotient is equal to 7 and when the dividend we find is exactly divisible by 3.

It can readily be seen that we thus find the sought-for number because if we carry its initial digit 7 to the end we obtain the new number which we have written as the quotient, that is a number which is three times as small as the sought-for number. In the above process every consecutive digit is determined uniquely by the digits found earlier, and therefore the number we obtained is the smallest of the numbers possessing the required properties. The calculations can be conveniently arranged as follows: in the upper line we write the digits of the dividend, in the second line we write the number for which every step of its division by 3 gives us the corresponding digit of the quotient and in the lower line we write the digits thus determined:

7	2	4	1	3	7	9	3	1	0	3	4	4	8	2	7	5	8	6	2
7	12	4	11	23	27	9	3	1	10	13	14	24	8	22	17	25	18	6	2
2	4	1	3	7	9	3	1	0	3	4	4	8	2	7	5	8	6	2	0
						0	6	8	9	6	5	5	1						
						20	26	28	19	16	15	5	21						
						6	8	9	6	5	5	1	7						

Thus, the smallest number possessing the required property is 7 241 379 310 344 827 586 206 896 551.

Third solution. By analogy with the first solution of Problem 34 (a), we obtain, using similar notation, the equality

$$(7 \cdot 10^m + X) \cdot \frac{1}{3} = 10X + 7$$

whence

$$X = \frac{7 \cdot 10^m - 21}{29}$$

The problem thus reduces to the determination of a number of the form 70 000 . . . whose division by 29 leaves a remainder of 21. Let the reader check that this procedure leads to the same result as in the first two solutions.

Remark. We can analogously solve the following problem:

It is required to find the smallest number with a given initial digit which decreases 3 times when the initial digit is carried to the end of the number. In order to include those solutions which begin with the digits 1 and 2 as well it is convenient to assume thas 0 can stand at the beginning of the numbers.

If the initial digit of a number is 0, it can readily be shown that only the number 0 posseses the required property. Let us write down the other (28-digit) numbers possessing this property:

1	034	482	758	620	689	655	172	413	793
2	068	965	517	241	379	310	344	827	586
3	103	448	275	862	068	965	517	241	379
4	137	931	034	482	758	620	689	655	172
5	172	413	793	103	448	275	862	068	965
6	206	896	551	724	137	931	034	482	758
8	275	862	068	965	517	241	379	310	344
9	310	344	827	586	206	896	551	724	137

In just the same way we can solve the following problem:

It is required to find the smallest whole number with a given initial digit a which decreases l times when this digit is carried to the end. It is also required to find all the numbers possessing the indicated property.

40. (a) By the condition of the problem, we have

$$\overline{xy\ \dots\ zt}\cdot a=\overline{tz\ \dots\ yx}$$

where a is one of the numbers 2, 3, 5, 6, 7 and 8 (the bars designate the numbers consisting of the corresponding digits).

If $a=5$ then x must be equal to 1 because, if otherwise, the number $\overline{xy\dots zt}\cdot 5$ would have more digits than the number $\overline{xy\dots zt}$ (we exclude the value $x=0$ because in this case $\overline{y\dots zt}=2\cdot\overline{tz\dots y}$, that is we arrive at the same problem with $a=2$). But the number $\overline{tz\dots y1}$ cannot be divisible by 5. In the same way we prove that a cannot be equal to 6 or 8.

If $a=7$ then x must also be equal to 1. But in this case t must be equal to 3 because, if otherwise, the number $\overline{1y\dots zt}\cdot 7$ cannot end with the digit 1. As to the equality $\overline{1y\dots z3}\cdot 7=\overline{3z\dots y1}$, it is quite obvious that it is inconsistent (because it is clear that the left-hand member of the equality is greater than the right-hand member).

If $a=2$ then x cannot be greater than 4. Since in this case the number $\overline{tz\dots yx}$ is even, we conclude that x must be equal to 2 or 4. For $x=4$ the digit t (the initial digit of the number $\overline{4y\dots zt}\cdot 2$) can only be equal to 8 or 9, and neither $\overline{4y\dots z8}\cdot 2$ nor $\overline{4y\dots z9}\cdot 2$ can have 4 as the last digit. If $x=2$ then t (the initial digit of the number $\overline{2y\dots zt}\cdot 2$) can only be equal to 4 or 5; but neither $\overline{2y\ \dots\ z4}\cdot 2$ nor $\overline{2y\ \dots\ z5}\cdot 2$ can end with 2.

Finally, if $a=3$, the digit x cannot be greater than 3. If $x=1$ then t must be equal to 7 (because the last digit of the number

$t \cdot 3$ is equal to 1). If $x = 2$ the digit t must be equal to 4 and if $x = 3$ the digit t must be equal to 1. But in the first case $\overline{tx \ldots yx}$ is greater than $\overline{xy \ldots zt} \cdot 3$, and in the second and in the third cases $\overline{tx \ldots yx}$ is less than $\overline{xy \ldots zt} \cdot 3$.

(b) Let $\overline{xy \ldots zt}$ be the sought-for number; then

$$\overline{xy \ldots zt} \cdot 4 = \overline{tz \ldots yx}$$

Since the number $\overline{xy \ldots zt} \cdot 4$ has the same number of digits as the number $\overline{xy \ldots zt}$, the digit x can be equal to 0, 1 or 2; since $\overline{tz \ldots yx}$ is divisible by 4, the digit x must be even. Consequently, x can only be equal to 0 or 2.

Let $x = 0$. It is evident that the number 0 possesses the required property. For the sake of convenience, we shall use decimal representations having one or more noughts at the beginning. Then we have $\overline{y \ldots zt} \cdot 4 = \overline{tz \ldots y0}$ whence $t = 0$ (since $t < 4$) and $\overline{y \ldots z} \cdot 4 = \overline{z \ldots y}$ because if a number possessing the required property begins with nought then its last digit is also equal to 0 and the number which is obtained when the first and the last noughts are deleted also possesses the required property.

Therefore it suffices to consider the value $x = 2$. In that case we have $\overline{2y \ldots zt} \cdot 4 = \overline{tz \ldots y2}$. Since $2 \cdot 4 = 8$, the digit t can only be equal to 8 or 9. However, the last digit of the product $t \cdot 4$ is 2; consequently, $t = 8$, that is we can write $\overline{2y \ldots z8} \cdot 4 = \overline{8z \ldots y2}$. Since $23 \cdot 4 > 90$, the digit y can only be equal to 0, 1 or 2. At the same time, the digit in the tens place of the product $\overline{z8} \cdot 4$ is odd for any z. Consequently, $y = 1$. Knowing the last two digits of the product $\overline{2y \ldots z8} \cdot 4$ we conclude that the last but one digit z of this number can only be equal to 2 or 7. Now, since $21 \cdot 4 > 82$, it follows that $z = 7$.

Thus, the sought-for number has the form $\overline{21 \ldots 78}$. If it has four digits we obtain the number 2178 satisfying the condition of the problem. Now let us consider the case when the number of digits of the sought-for number exceeds 4. In that case we have

$$\overline{21uv \ldots rs78} \cdot 4 = \overline{87sr \ldots vu12}$$

whence, after simple transformations, we obtain

$$84 \cdot 10^{k+2} + 312 + \overline{uv \ldots rs00} \cdot 4 = 87 \cdot 10^{k+2} + 12 + \overline{sr \ldots vu00}$$

and

$$\overline{uv \ldots rs} \cdot 4 + 3 = \overline{3sr \ldots vu}$$

Since the product of the number $\overline{uv \ldots rs}$ by 4 has more digits than the given number itself and since this product has the initial digit 3 (or begins with the combination of the digits 29) we see that u can only be equal to 9, 8, or 7. Further, since $\overline{3sr \dots vu}$ is an odd number, u can only be equal to 9 or 7. Let us consider separately these two possibilities.

1°. $u = 9$. In this case we obviously have

$$\overline{9v \dots rs} \cdot 4 + 3 = \overline{3sr \dots v9}$$

whence it follows that $s = 9$ (because $s \cdot 4$ ends with 6; if $s = 4$ then $\overline{34r \dots v9}$ is less than $\overline{9v \dots r4} \cdot 4 + 3$) and

$$\overline{9v \dots r9} \cdot 4 + 3 = \overline{39r \dots v9} \quad \text{and} \quad \overline{v \dots r} \cdot 4 + 3 = \overline{3r \dots v}$$

Thus, the number obtained from the number $\overline{uv \dots rs}$ by discarding the digits 9 standing at the beginning and at the end possesses the same property as the number $\overline{uv \dots rs}$ itself. In particular, $\overline{uv \dots rs}$ can be equal to 9; 99; 999 etc.; in this way we obtain the numbers

$$21\,978 \quad 219\,978; \quad 2\,199\,978; \dots$$

satisfying the conditions of the problem.

2°. $u = 7$. In this case we have

$$\overline{7v \dots rs} \cdot 4 + 3 = \overline{3sr \dots v7}$$

whence, by analogy with the argument at the beginning of the solution of the problem, we readily find that $s = 1$, $v = 8$ and $r = 2$, that is the number $\overline{uv \dots rs}$ is of the form $\overline{78 \dots 21}$ and the number obtained from $\overline{uv \dots rs}$ by discarding the combinations of the digits 78 and 21 at the beginning and at the end respectively is 4 times as small as its reversion.

It follows that if a number which is 4 times as small as its reversion differs from the numbers in the sequence

$$0;\ 2178;\ 21\,978;\ 219\,978;\ \dots;\ 21\underbrace{99\dots9}_{k\ \text{digits}}78;\ 21\underbrace{99\dots99}_{(k+1)\ \text{digits}}78;\ \dots \quad (*)$$

then there are the same combinations of digits at the beginning and at the end of this number and these digits form one of the numbers belonging to the above sequence; besides, if these combinations of digits are deleted (both at the beginning and at the end) then we also obtain a number which is 4 times as small as its reversion or, as in the case of the number 21 782 178, all the digits of the number turn out to be deleted.

Therefore the decimal representation of any number which is 4 times as small as its reversion must have the form

$$P_1P_2 \dots P_{n-1}P_nP_{n-1} \dots P_2P_1$$

or the form

$$P_1P_2 \ldots P_{n-1}P_nP_nP_{n-1} \ldots P_2P_1$$

where P_1, P_2, ..., P_n are combinations of digits forming some of the numbers belonging to sequence (*). For instance, such are the numbers

$$2\,197\,821\,978; \quad 2\,199\,782\,178\,219\,978;$$

$$21\,978\,021\,997\,800\,219\,978\,021\,978 \quad \text{and} \quad 02\,199\,999\,780$$

(the last of these numbers can also be regarded as the solution of the problem on condition that we are allowed to write 0 at the beginning of the decimal representation of a number).

We can similarly prove that all the numbers which are 9 times as small as their reversions are obtained from the numbers forming the sequence

$$0;\ 1089;\ 10\,989;\ 109\,989;\ \ldots;\ 10\underbrace{99\ldots9}_{k \text{ times}}89\ \ldots;\ 10\underbrace{99\ldots99}_{(k+1) \text{ times}}89;\ \ldots$$

in the same way as the numbers which are 4 times as small as their reversions are obtained from the numbers belonging to sequence (*).

41. **(a)** Let us denote by p the number consisting of the first three digits of the sought-for number N and by q the number consisting of the last three digits of N. Then the condition of the problem implies

$$1000q + p = 6\,(1000p + q) = 6N$$

whence

$$(1000q + p) - (1000p + q) = 999\,(q - p) = 5N$$

which means that N is divisible by 999.

Further, we have $p + q = (1000p + q) - 999p = N - 999p$ whence it follows that $p + q$ is also divisible by 999. On the other hand, p and q are three-digit numbers which obviously cannot be equal to 999 simultaneously, and consequently

$$p + q = 999$$

Now we readily find that

$$(1000q + p) + (1000p + q) = 1001\,(p + q) = 7N$$

and, consequently,

$$7N = 999\,999 \quad \text{and} \quad N = 142\,857$$

(b) By analogy with the solution of Problem 41 (a), on denoting as p and q the numbers formed of the first four and of the last four digits of the sought-for number N respectively, we can write

$$7N = 10\,001\,(p+q) = 99\,999\,999$$

It is evident that this relation cannot hold for any integral number N (because 99 999 999 is not divisible by 7).

42. Let x be a number satisfying the condition of the problem. Since both $6x$ and x are six-digit numbers, the initial digit is the decimal representation of the number x is equal to 1. Therefore we conclude that

(1) the initial digits of the decimal representations of the numbers x, $2x$, $3x$, $4x$, $5x$ and $6x$ are all different, and consequently they form the whole set of digits contained in the decimal representation of the number x;

(2) all the digits in the decimal representation of the number x are different from one another.

The set of these digits does not contain 0 and therefore the last digit of the number x is odd (if otherwise, $5x$ would end with nought) and differs from 5 (because, if otherwise, the last digit of $2x$ would be 0). Therefore the last digits in the decimal representations of the numbers x, $2x$, $3x$, $4x$, $5x$ and $6x$ are all different, and hence they also form the whole set of the digits contained in the decimal representation of the number x. Consequently, this set contains 1. The digit 1 can only be the last digit of the number $3x$ because $2x$, $4x$, and $6x$ end with even digits and $5x$ ends with 5 and the decimal representation of the number x involves one digit 1 which is its initial digit. Thus, the number x ends with the digit 7, the number $2x$ with the digit 4, the number $3x$ with the digit 1, the number $4x$ with the digit 8, the number $5x$ with the digit 5 and the number $6x$ with the digit 2. Since the first digits of these numbers belong to the same set of digits but are arranged in the increasing order, we can write

$$x \cdot 1 = 1****7$$
$$x \cdot 2 = 2****4$$
$$x \cdot 3 = 4****1$$
$$x \cdot 4 = 5****8$$
$$x \cdot 5 = 7****5$$
$$x \cdot 6 = 8****2$$

where the stars stand in the places occupied by the unknown digits.

Now we note that in the table we have written not only every line contains the six different digits 1, 2, 4, 5, 7, and 8 arranged in a certain order but also every column consists of the same six *different* digits arranged in some order. Indeed, let us suppose that, for instance, the third digits of the numbers $x \cdot 2$ and $x \cdot 5$ coincide and are equal to a (a can assume one of the two values not equal to the first and to the last digits of the two numbers in question). Then the difference $x \cdot 5 - x \cdot 2 = x \cdot 3$ is a six-digit number the third digit of whose decimal representation is 0 or 9 (this follows from the rule according to which the subtraction of numbers written as a column is carried out). But this conclusion cannot be true because, as we know, the decimal representation of the number $x \cdot 3$ involves the digits 1, 2, 4, 5, 7 and 8.

Now let us again write as a column the numbers $x \cdot 1$, $x \cdot 2$, $x \cdot 3$, $x \cdot 4$, $x \cdot 5$ and $x \cdot 6$ in order to add them together:

$$x \cdot 1 = 1 **** 7$$
$$x \cdot 2 = 2 **** 4$$
$$x \cdot 3 = 4 **** 1$$
$$x \cdot 4 = 5 **** 8$$
$$x \cdot 5 = 7 **** 5$$
$$x \cdot 6 = 8 **** 2$$

Taking into account that the sum of the digits of every column is equal to $1 + 2 + 4 + 5 + 7 + 8 = 27$ we get

$$x \cdot 21 = 2\,999\,997$$

whence $x = 142\,857$. The number x we have is nothing but the sought-for number; indeed, it is readily seen that

$$x = 142\,857$$
$$2x = 285\,714$$
$$3x = 428\,571$$
$$4x = 571\,428$$
$$5x = 714\,285$$
$$6x = 857\,142$$

43. Let $N = \overline{xyz} = 100x + 10y + z$ be the sought-for number, the symbols x, y and z designating its digits. The permutations of the digits of the number N give us the new numbers $N_1 = \overline{yxz} = 100y + 10x + z, \ldots, N_5 = \overline{zyx} = 100z + 10y + x$. The sum $N + N_1 + \ldots + N_5$ of all these numbers must be equal to the product of the number N by 6 whence we readily obtain

$$(2 \cdot 100 + 2 \cdot 10 + 2)(x + y + z) = 6(100x + 10y + z)$$

(because in the 6-tuple of the numbers N, N_1, N_2, N_3, N_4, N_5 each of the digits, for instance x, is encountered twice in the ones place, twice in the tens place and twice in the hundreds place). This means that

$$37(x+y+z)=100x+10y+z$$

whence

$$63x=27y+36z$$

On cancelling by 9 we find

$$7x=3y+4z; \quad \text{that is} \quad 7(x-y)=4(z-y)$$

Now, since the absolute values of the differences $x-y$ and $z-y$ do not exceed 9, it follows that the last equality can only hold when $x-y=0$, $z-y=0$ or $x-y=4$, $z-y=7$ or $x-y=-4$, $z-y=-7$. If $z-y=7$ then $x=9$; 8 or 7 and if $z-y=-7$ then $y=9$; 8 or 7. Therefore we obtain the following 15 possible values of the number N:

$$N=111;\ 222;\ 333;\ 444;\ 555;\ 666;\ 777;\ 888;\ 999;\ 407;\ 518;$$
$$629;\ 370;\ 481 \quad \text{or} \quad 592$$

(the "solution" $N=000$ has been discarded).

44. It is clear that A and A' must be 10-digit numbers. Let $A=\overline{a_{10}a_9a_8\ldots a_1}$ and $A'=\overline{a'_{10}a'_9a'_8\ldots a'_1}$ (here a_{10}, a_9, a_8, ..., a_1 are the consecutive digits of the number A and a'_{10}, a'_9, ..., a'_1 are the digits of the number A'). Suppose that we write the numbers A and A' as a column to add them together. It is clear that their sum can be equal to the number 10 000 000 000 only in the case when there is such an index i (where $0 \leqslant i \leqslant 9$) for which $a_1+a'_1=0$, $a_2+a'_2=0$, ..., $a_i+a'_i=0$,

$$a_{i+1}+a'_{i+1}=10, \quad a_{i+2}+a'_{i+2}=9, \ \ldots, \ a_{10}+a'_{10}=9 \qquad (*)$$

(if $i=9$ then there are no sums of the form $a_{i+2}+a'_{i+2}$, $a_{i+3}+a'_{i+3}$, ... which are equal to nine and if $i=0$ then there are no sums of the form $a_1+a'_1$, ..., $a_i+a'_i$ which are equal to zero). On adding together all sums (*) we obtain

$$(a_1+a'_1)+(a_2+a'_2)+ \ \ldots \ +(a_{10}+a'_{10})=10+9(9-i)$$

Since a_1a_2, ..., a_{10} and $a'_1a'_2$, ..., a'_{10} are sequences consisting of *the same* digits but arranged in different order we conclude that the right-hand member of the last equality is an even number equal to $2(a_1+a_2+\ldots+a_{10})$; therefore the number $10+9(9-i)$ is also *even*. It follows that the index i must necessarily be *odd*, that is i cannot be equal to zero (there must be $i=1$ or $i=3$ or $i=5$, ...). Hence, $a_1+a'_1=0$, which implies that $a_1=a'_1=0$. The last equalities show that both numbers A and A' are divisible by 10.

45. Let us write the numbers M and N as a column to add them together in accordance with the ordinary rule of arithmetic. If we suppose that all the digits of the resultant sum $M+N$ are *odd* then the sum of the last digits is odd, which implies that the sum of the initial digits is also odd (the columns consisting of the initial digits and of the last digits differ only in the order in which the digits are written). This is only possible if after the addition of the digits in the 2nd column unity is not carried from that column to the 1st column, which means that the sum of the digits of the 2nd column is less than 10 and, consequently, so is the sum of the digits of the last but one column. Therefore unity is not carried from the last but one column to the 3rd (counting off from right to left) column either because the case when the sum of the digits of the last but one column is equal to $9 < 10$ and unity is carried from that column to the next one because it is taken from the last column is impossible. For, in this case, the last but one digit of the sum $M+N$ must be equal to 0, that is it must be *even*, which contradicts the hypothesis. Thus, in the addition process *the digits in the last two* (*and in the first two*) *columns do not affect the other digits of the sum* $M+N$. Therefore we can simply discard the first two and the last two digits in the numbers M and N and continue the argument for the corresponding "truncated" (13-digit) numbers M_1 and N_1.

Now let us consider the sum M_1+N_1 of the numbers M_1 and N_1; as before, it can be shown that *if all the digits of the number* M_1+N_1 *are odd then when we write the* 13-*digit numbers* M_1 *and* N_1 *as a column to add them together the first two digits and the last two digits of the numbers* M_1 *and* N_1 *do not affect the other digits of the sum* M_1+N_1 (that is they do not affect the digits of the sum M_1+N_1 except the first two and the last two digits). This means that we can "truncate" the numbers M_1 and N_1 by discarding in each of them the first two and the last two digits and pass to the corresponding 9-digit numbers M_2 and N_2. Next we perform the same operation on the numbers M_2 and N_2 and pass to the corresponding 5-digit numbers M_3 and N_3; finally, in just the same way we pass from the numbers M_3 and N_3 to the corresponding "truncated" (one-digit!) numbers M_4 and N_4 which are equal to the digits of the numbers M and N standing at the middle of the decimal representations of M and N, these digits being coincident. It is clear that the digits (or, more precisely, one digit!) of the number $M_4+N_4=2M_4$ cannot be odd (because the number $2M_4$ is even!), whence we conclude that all the digits of the sum $M+N$ cannot be odd either.

Remark. It is clear that the above argument is of a general character and remains applicable to any two numbers M and N (which are written with the aid of the same digits but taken in "reverse" order) provided that the number of the digits in each of them has the form $4n+1$, that is provided that the division of the number of the digits by 4 leaves a remainder of 1. But if the number of the digits of the number M (and of the "reverted" number N) does not have the form $4n+1$, then it may happen that the sum $M+N$ is written only with the aid of odd digits (let the reader prove this).

46. (a) We have $n^3-n=(n-1)n(n+1)$, and one of the three consecutive whole numbers in the product on the right-hand side must necessarily be divisible by 3.

(b) We have $n^5-n=n(n-1)(n+1)(n^2+1)$. If the whole number n ends with one of the digits 0, 1, 4, 5, 6 or 9 then one of the first three factors on the right-hand side is divisible by 5. If n ends with one of the digits 2, 3, 7 or 8 then the last digit of n^2 is 4 or 9, and n^2+1 is divisible by 5.

(c) We have $n^7-n=n(n-1)(n+1)(n^2-n+1)(n^2+n+1)$. If n is divisible by 7 or if the division of n by 7 leaves a remainder equal to 1 or 6 then one of the first three factors on the right-hand side is divisible by 7. If the division of n by 7 leaves a remainder equal to 2 (that is $n=7k+2$) then the division of n^2 by 7 leaves a remainder of 4 (because $n^2=49k^2+28k+4$), and consequently n^2+n+1 is divisible by 7. In the same way we can prove that if the remainder resulting from the division of n by 7 is equal to 4 then n^2+n+1 is exactly divisible by 7 and if the division of n by 7 leaves a remainder equal to 3 or 5 then n^2-n+1 is divisible by 7.

(d) We have $n^{11}-n=n(n-1)(n+1)(n^8+n^6+n^4+n^2+1)$. If n is divisible by 11 or if the division of n by 11 leaves a remainder equal to 1 or 10 then one of the first three factors on the right-hand side is divisible by 11. If the remainder resulting from the division of n by 11 is equal to 2 or 9 (that is if $n=11k\pm 2$) then the remainder resulting from the division of n^2 by 11 is equal to 4 (because $n^2=121k^2\pm 44k+4$), the division of n^4 by 11 results in the remainder equal to $5=16-11$, the division of n^6 by 11 leaves a remainder equal to $9=20-11$, (because $n^6=n^4\cdot n^2=(11k_1+5)(11k_2+4)=121k_1k_2+11(4k_1+5k_2)+20$) and the division of n^8 by 11 leaves $3=25-22$ in the remainder. It follows that $n^8+n^6+n^4+n^2+1$ is divisible by 11. It can similarly be shown that $n^8+n^6+n^4+n^2+1$ is divisible by 11 if the remainder resulting from the division of n by 11 is equal to ± 3, ± 4 or ± 5.

(e) We have $n^{13}-n=n(n-1)(n+1)(n^2+1)(n^4-n^2+1)\times\times(n^4+n^2+1)$. By analogy with the solutions of the foregoing problems, we conclude that if n is divisible by 13 or if the division of n by 13 leaves a remainder equal to $+1$ or -1 then one of the first three factors on the right-hand side is divisible by 13, if the remainder resulting from the division of n by 13 is equal to ± 5 then n^2+1 is divisible by 13, if the remainder resulting from the division of n by 13 is equal to ± 2 or ± 6 then $n^4 - n^2 + 1$ is divisible by 13 and if the remainder resulting from the division of n by 13 is equal to ± 3 or ± 4 then $n^4 + n^2 + 1$ is divisible by 13.

47. **(a)** The difference of two powers with equal even exponents is exactly divisible by the sum of the bases; therefore $3^{6n} - 2^{6n} = 27^{2n} - 8^{2n}$ is divisible by $27 + 8 = 35$.

(b) It can easily be verified that

$$n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)n(n + 1)(n + 2)$$

Here there are five consecutive whole numbers in the product on the right-hand side one of which must necessarily be divisible by 5; besides, at least one of the factors is divisible by 3 and at least two of them are divisible by 2; further, at least one of the last two factors must also be divisible by 4. Thus, the product of five consecutive whole numbers is always divisible by $5 \cdot 3 \cdot 2 \cdot 4 = 120$ (cf. the solution of Problem 46 (a)).

(c) Let us make use of the identity

$$n^2 + 3n + 5 \equiv (n + 7)(n - 4) + 33$$

For this expression to be divisible by 11 it is necessary that $(n + 7)(n - 4)$ should be divisible by 11. Since we have $(n + 7) - (n - 4) = 11$, both factors $n + 7$ and $n - 4$ should be simultaneously divisible or not divisible by 11. Therefore if the number $(n + 7)(n - 4)$ is divisible by 11 then it is also divisible by 121 and, consequently, $(n + 7)(n - 4) + 33$ cannot be divisible by 121.

48. **(a)** It can readily be checked that

$$56\,786\,730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$$

and hence it only remains to prove that the given expression is divisible by each of the prime factors on the right-hand side. If both m and n are odd numbers then the number $m^{60} - n^{60}$ is even; consequently, $mn(m^{60} - n^{60})$ must necessarily be even (that is it must be divisible by 2). Further, from the result of Problem 46 it follows that if k is equal to 3, 5, 7, 11 or 13 and if n is not divisible by k then the difference $n^{k-1} - 1$ must necessarily be divisible by k. In particular, it follows that if both m and n are not divisible by 3 then the numbers $m^{60} - 1 = (m^{30})^2 - 1$ and $n^{60} - 1 = (n^{30})^2 - 1$ are divisible by 3, that is the division of m^{60} and of n^{60} by 3 leaves one and the same remainder equal to 1. Consequently, if mn is not divisible by 3 then $m^{60} - n^{60}$ is divisible by 3, whence it follows that in all the cases the product $mn(m^{60} - n^{60})$ is divisible by 3. In just the same way we can prove that the difference

$$m^{60} - n^{60} = (m^{15})^4 - (n^{15})^4 = (m^{10})^6 - (n^{10})^6 =$$
$$= (m^6)^{10} - (n^6)^{10} = (m^5)^{12} - (n^5)^{12}$$

is divisible by 5 when neither m nor n is divisible by 5, is divisible by 7 when neither m nor n is divisible by 7, is divisible by 11 when neither m nor n is divisible by 11 and is divisible by 13 when neither m nor n is divisible by 13. We have thus proved that $mn(m^{60} - n^{60})$ is always divisible by $2\cdot3\cdot5\cdot7\cdot11\cdot13$.

It can similarly be shown that the expression $mn(m^{60} - n^{60})$ is divisible by 31 and by 61 (because for any integral n the expression $n^{31} - n$ is divisible by 31 and the expression $n^{61} - n$ is divisible by 61; see Problem 340 below).

(b) Let us represent the given expression in the form

$$(m - 2n)(m - n)(m + n)(m + 2n)(m + 3n)$$

For $n \neq 0$ all the five factors of this product are pairwise different. At the same time, the number 33 cannot be factored as a product of more than four different integers (the factorization into four such factors can be performed in several ways, for instance, $33 = (-11)\cdot3\cdot1\cdot(-1)$ or $33 = 11\cdot(-3)\cdot1\cdot(-1)$).

In the case when $n = 0$ the given expression turns into m^5 and cannot be equal to 33 for any integral m.

49. First of all we note that $323 = 17\cdot19$; hence we have to establish the condition under which the number N indicated in the problem is divisible both by 17 and by 19. Let us begin with the case when n is an *even* number: $n = 2k$. It is clear that $20^n - 3^n$ is divisible by $20 - 3 = 17$ for all n; on the other hand, $16^n - 1^n = 16^{2k} - 1^{2k}$ is divisible by $16^2 - 1^2 = (16 - 1)(16 + 1) = 15\cdot17$ and, consequently, for even n the number $16^n - 1$ is also divisible by 17, that is in this case $N = (20^n - 3^n) + (16^n - 1)$ is divisible by 17. Further, the number $20^n - 1$ is divisible by $20 - 1 = 19$ for all n and the number $16^n - 3^n = 16^{2k} - 3^{2k}$ is divisible by $16^2 - 3^2 = (16 - 3)(16 + 3) = 19\cdot13$, that is it is also divisible by 19; therefore the number $N = (20^n - 1) + (16^n - 3^n)$ is divisible by 19. Thus, *the number N is divisible by* 323 *for all even values of n*. If the number n is *odd*, that is $n = 2k + 1$, then the difference $20^n - 3^n$ is again divisible by 17. Since $16^{2k} - 1$ is divisible by 17, the division of 16^{2k} by 17 leaves a remainder equal to 1, and consequently the division of the number $16^{2k+1} = 16^{2k}\cdot16$ by 17 leaves a remainder equal to $1\cdot16 = 16$. Therefore the number $16^n - 1 = 16^{2k+1} - 1$ *is not divisible* by 17 (its division by 17 results in the remainder equal to 15). Hence, *for any odd value of n the number N is not divisible by* 17 *and therefore it cannot be divisible by* 323.

Thus, the number N is divisible by 323 if and only if n is *even.*

50. If the last digit of the number n is 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 then the last digit of n^2 is 0, 1, 4, 9, 6, 5, 6, 9, 4 or 1 respectively, and consequently the number n^2+n ends with 0, 2, 6, 2, 0, 0, 2, 6, 2 or 0 respectively and the number n^2+n+1 ends with the digit 1, 3, 7, 3, 1, 1, 3, 7, 3 or 1 respectively. Thus, the number n^2+n+1 cannot have 0 or 5 as its last digit, that is it cannot be divisible by 5 (and consequently it *cannot be divisible by* 1955 either).

51. Any whole number is either divisible by 5 or can be written in one of the following four forms: $5k+1$, $5k+2$, $5k-2$ and $5k-1$. If a number is divisible by 5 then its hundredth power is obviously divisible by $5^3=125$. Further, by Newton's binomial formula, we obtain

$$(5k\pm 1)^{100}=(5k)^{100}\pm \dots +\frac{100\cdot 99}{1\cdot 2}(5k)^2\pm 100\cdot 5k+1$$

where all the terms marked by the dots contain the factor $5k$ to the power not less than 3, and consequently they are all divisible by 125. Analogously,

$$(5k\pm 2)^{100}=(5k)^{100}\pm \dots +\frac{100\cdot 99}{1\cdot 2}(5k)^2\cdot 2^{98}\pm 100\cdot 5k\cdot 2^{99}+2^{100}$$

The numbers $\frac{100\cdot 99}{1\cdot 2}(5k)^2=125\cdot 990k^2$ and $100\cdot 5k=125\cdot 4k$ are divisible by 125. As to the number 2^{100}, it can be represented in the form

$$(5-1)^{50}=5^{50}- \dots +\frac{50\cdot 49}{1\cdot 2}\cdot 5-50\cdot 5+1$$

whence we readily see that the division of this number by 125 leaves a remainder equal to 1.

Thus, the hundredth power of a number divisible by 5 must be divisible by 125, and the division by 125 of the hundredth power of a number not divisible by 5 leaves a remainder 1.

52. We have to prove that if N is relatively prime to 10 then $N^{101}-N=N(N^{100}-1)$ is divisible by 1000, that is we must prove that $N^{100}-1$ is divisible by 1000. First of all, it is quite clear that if N is an odd number then $N^{100}-1=(N^{50}+1)\times(N^{25}+1)(N^{25}-1)$ is divisible by 8. Further, from the result of the foregoing problem it follows that if N is not divisible by 5 then $N^{100}-1$ is divisible by 125. Thus, we see that $N^{100}-1$ is divisible by $8\cdot 125=1000$ for N relatively prime to 10.

53. Let N be the sought-for number; then N^2-N has three noughts at the end, that is this difference is divisible by 1000. Since $N^2-N=N(N-1)$ and since N and $N-1$ are relatively prime numbers, this can only be possible when one of these numbers is divisible by 8 while the other is divisible by 125 (neither of these numbers itself is divisible by 1000 because N is a three-digit number).

If N is a three-digit number divisible by 125 then $N-1$ can be divisible by 8 only when $N = 625$ (in that case $N-1 = 624$), which can easily be verified. It can also be easily shown that it $N-1$ is a three-digit number divisible by 125 then N is divisible by 8 only when $N-1 = 375$, that is in this case $N = 376$.

Now we note that since $N^{k-1}-1$ is exactly divisible by $N-1$ for any integral $k \geqslant 2$, the number $N^k - N = N(N^{k-1}-1)$ is divisible by $N(N-1) = N^2 - N$ for any integral k. Therefore if the last three digits of $N^2 - N$ are noughts then $N^k - N$ also has three noughts at the end for any integral $k \geqslant 2$, that is N^k ends with the same three digits as N. It follows that the numbers 625 and 376 (and only these numbers) satisfy the conditions of the problem.

54. Let us find the last two digits of the number N^{20}. The number N^{20} is divisible by 4 because N is even. Further, the number N is not divisible by 5 (if it were divisible by 5 then it would also be divisible by 10), and consequently N can be represented in the form $5k \pm 1$ or $5k \pm 2$ (cf. the solution of Problem 51). The division of the number

$$(5k \pm 1)^{20} = (5k)^{20} \pm 20\,(5k)^{19} + \ldots + \frac{20 \cdot 19}{1 \cdot 2}(5k)^2 \pm 20 \cdot 5k + 1$$

by 25 leaves a remainder of 1 while the division of the number

$$(5k \pm 2)^{20} = (5k)^{20} \pm 20\,(5k)^{19} \cdot 2 + \ldots$$
$$\ldots + \frac{20 \cdot 19}{1 \cdot 2}(5k)^2 \cdot 2^{18} \pm 20 \cdot 5k \cdot 2^{19} + 2^{20}$$

by 25 leaves the same remainder as the division of the number $2^{20} = (2^{10})^2 = (1024)^2 = (1025-1)^2$, that is 1. The fact that the remainder resulting from the division of the number N^{20} by 25 is equal to 1 implies that the last two digits of this number can only be 01; 26; 51 or 76. Besides, taking into account that N^{20} must be divisible by 4, we conclude that the last two digits of this number can only be 76. Thus the digit in the tens place of the number N^{20} is 7.

Now let us determine the last three digits of the number N^{200}. The number N^{200} is divisible by 8. Further, since N is relatively prime to 5, the division of N^{100} by 125 leaves a remainder equal to 1 (see the solution of Problem 51): $N^{100} = 125k + 1$. Therefore the division of the number $N^{200} = (125k+1)^2 = 125^2k^2 + 250k + 1$ by 125 also leaves a remainder 1. Consequently, the last three digits of N^{200} can be 126; 251; 376; 501; 626; 751 or 876. However, the number N^{200} is divisible by 8 and therefore it must end with the digits 376. Thus, the digit in the hundreds place of the number N^{200} is equal to 3.

Remark. It can easily be seen that not only N^{200} but also the number N^{100} must necessarily have the digits 376 at the end.

55. The sum $1+2+3+\ldots+n$ is equal to $n(n+1)/2$; consequently, we have to prove that if k is odd then $S_k = 1^k + 2^k + 3^k + \ldots + n^k$ is divisible by $n(n+1)/2$.

First of all we should take into account that $a^k + b^k$ is divisible by $a+b$ for any odd k. Let us consider separately the following two cases:

A. The number n is *even*. Then the sum S_k is divisible by $n+1$ because each of the sums

$$1^k+n^k,\ \ 2^k+(n-1)^k,\ \ 3^k+(n-2)^k,\ \ldots,\ \left(\frac{n}{2}\right)^k+\left(\frac{n}{2}+1\right)^k$$

is divisible by

$$1+n=2+(n-1)=3+(n-2)=\ldots=\frac{n}{2}+\left(\frac{n}{2}+1\right)$$

The sum S_k is also divisible by $n/2$ because the expressions $1^k+(n-1)^k$, $2^k+(n-2)^k$, $3^k+(n-3)^k, \ldots, \left(\frac{n}{2}-1\right)^k+\left(\frac{n}{2}+1\right)^k$, $\left(\frac{n}{2}\right)^k$, n^k are all divisible by $n/2$.

B. The number n is *odd*. In this case the sum S_k is divisible by $(n+1)/2$ because the expressions 1^k+n^k, $2^k+(n-1)^k$, $3^k+(n-2)^k$, $\ldots$, $\left(\frac{n-1}{2}\right)^k+\left(\frac{n+3}{2}\right)^k$ and $\left(\frac{n+1}{2}\right)^k$ are all divisible by $(n+1)/2$. The sum S_k is also divisible by n since the numbers $1^k+(n-1)^k$, $2^k+(n-2)^k$, $3^k+(n-3)^k$, $\ldots$, $\left(\frac{n-1}{2}\right)^k+\left(\frac{n+1}{2}\right)^k$ and n^k are all divisible by n.

56. Let

$$N=a_n\cdot 10^n+a_{n-1}\cdot 10^{n-1}+a_{n-2}\cdot 10^{n-2}+\ldots+a_1\cdot 10+a_0$$

be the given number where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are its digits which can assume the values $0, 1, 2, \ldots, 9$.

Let us subtract from N the number

$$M=a_0-a_1+a_2-a_3+\ldots\pm a_n$$

equal to *the algebraic sum of the digits of the number N taken with the alternating signs "+" and "−"*. On grouping the terms in the appropriate manner we obtain the expression

$$N-M=a_1(10+1)+a_2(10^2-1)+a_3(10^3+1)+$$
$$+a_4(10^4-1)+\ldots+a_n(10^n\pm 1)$$

which is exactly divisible by 11 since each of its addends is divisible by 11 (because from the well-known fact that to the multiplication of numbers there corresponds the multiplication of the remainders obtained when these numbers are divided by a given number it readily follows that when

$$10^k = (11-1)^k$$

is divided by 11 the remainder is equal to -1 for odd k and to $+1$ for even k). Thus, the difference $N - M$ is divisible by 11, that is the numbers N and M are simultaneously divisible or not divisible by 11.

57. The division of the number 15 by 7 leaves a remainder 1. It follows that

$$15^2 = (7\cdot 2+1)(7\cdot 2+1) = 7n_1 + 1$$

and therefore the division of 15^2 by 7 also leaves a remainder equal to 1. Similarly,

$$15^3 = 15^2\cdot 15 = (7n_1+1)\cdot(7\cdot 2+1) = 7n_2 + 1$$

whence it follows that the division of 15^3 by 7 also leaves 1 and so on, that is, generally, the division of any power of the number 15 by 7 leaves a remainder equal to 1. Now, on subtracing the sum $1+2+3+4+\ldots+14 = 105$ from the given number and grouping the terms in the appropriate manner we obtain the number

$$13(15-1)+12(15^2-1)+11(15^3-1)+\ldots$$
$$\ldots+2(15^{12}-1)+1(15^{13}-1)$$

which is exactly divisible by 7. Since the difference between the given number and the number $105 = 7\cdot 15$ is divisible by 7, it follows that the original number is also divisible by 7.

58. Let K be an n-digit number. Among the $(n+2)$-digit numbers whose first two digits are 1 and 0 (that is among the numbers of the form $\overline{10a_1a_2\ldots a_n}$ (where 1, 0, $a_1, \ldots, a_n$ are the digits of the number and the bar designates the member itself) there always exists at least one number divisible by K. Let $\overline{10b_1b_2\ldots b_n}$ be such a number. Then, by the condition of the problem, both numbers $\overline{b_1b_2\ldots b_n10}$ and $\overline{b_1b_2\ldots b_n01}$ are divisible by K. Their difference is equal to 9 and it is also divisible by K. The only divisors of 9 are the numbers 1, 3 and 9, whence follows the assertion of the problem.

59. It is clear that $d = \underbrace{333\ldots33}_{100\text{ threes}} = 3\cdot\underbrace{111\ldots11}_{100\text{ ones}} = 3n;$

therefore the sought-for number $N = \underbrace{1111 \ldots 111}_{k \text{ ones}}$ must be exactly divisible by the numbers n and 3 (n is not divisible by 3 because the sum of the digits of the number n is equal to 100 and is not divisible by 3). If k is a number of the form $k = 100q + r$ where $r < 100$ (but $r \geqslant 0$) then, obviously, $N = \underbrace{11 \ldots 11}_{100q \text{ ones}}\underbrace{00 \ldots 00}_{r \text{ noughts}} + \underbrace{11 \ldots 11}_{r \text{ ones}} = M + R$ where $R = \underbrace{11 \ldots 11}_{r \text{ ones}}$ and $M = \underbrace{11 \ldots 11}_{100q \text{ ones}}\underbrace{00 \ldots 00}_{r \text{ noughts}}$ the number M being divisible by n (the divisibility of M by n becomes quite obvious if we consider the process of long division of M by n). Thus, N is divisible by n if and only if $R = 0$, that is if and only if $r = 0$ and, consequently, if and only if k is divisible by 100.

Now, if $k = 100q$ then the sum of the digits of the number N is equal to $100q$; this sum is divisible by 3 (and, consequently, the number N is also divisible by 3) if and only if q is divisible by 3. Therefore the smallest number $N = \underbrace{111 \ldots 11}_{k \text{ ones}}$ divisible by d consists of 300 ones.

60. Since a is obviously an even number, it only remains to prove that the product aA is divisible by 3. The last digits of the numbers $2^{k+1} = 2N$ and $2a$ (where a is the last digit of N) coincide. Therefore, on multiplying consecutively the powers of 2 again by 2 (that is on increasing consecutively the exponents of the powers of 2) we find that the last digits of the numbers $2^1 = 2$; $2^2 = 4$; $2^3 = 8$; $2^4 = 16$; $2^5 = 32$; ... form the following sequence of periodically alternating digits:

$$2;\ 4;\ 8;\ 6;\quad 2;\ 4;\ 8;\ 6;\quad 2;\ 4;\ 8;\ 6;\quad \ldots$$

On the other hand if the division of $2^k = N$ by 3 leaves a remainder equal to 1 then the division of the number $2^{k+1} = 2N$ by 3 leaves a remainder of 2, and if $N = 3l + 2$ then the number $2^{k+1} = 2N$ has the form $3 \cdot (2l) + 4 = 3 \cdot (2l + 1) + 1$ and therefore its division by 3 leaves a remainder 1. Consequently, when the numbers belonging to the same sequence of the powers of two are divided by 3 we obtain the following sequence of periodically alternating remainders:

$$2;\ 1;\ 2;\ 1;\quad 2;\ 1;\ 2;\ 1;\quad 2;\ 1;\ 2;\ 1;\quad \ldots$$

Thus, if the last digit of the number N is equal to 2 or 8 (that is if $a = 2$ or $a = 8$) then the division of N by 3 leaves a remainder of 2, and if $a = 4$ then the remainder resulting from the division of N by 3 is equal to 1. (The case when $a = 6$ is of no interest because for $a = 6$ the product $aA = 6A$ must necessarily be divisible by 6.) It follows that in all these three cases the number $N - a = 10A$ is divisible by 3, and consequently the number A is also divisible by 3. Hence, the product aA is divisible by 6 for all $k \geqslant 1$; for k equal to 1, 2 or 3 this conclusion is quite trivial because in these cases $A = 0$ and the number $aA = 0$ is divisible by *any* number.

61. We have to show that the number

$$N = 27\,195^8 - 10\,887^8 + 10\,152^8$$

is exactly divisible by $26\,460 = 2^2 \cdot 3^3 \cdot 5 \cdot 7^2$. The proof consists of the following two stages.

1°. We have $N = 27\,195^8 - (10\,887^8 - 10\,152^8)$. The number 27 195 is equal to the product $3 \cdot 5 \cdot 7^2 \cdot 37$, and consequently the number 27 195 is divisible by $5 \cdot 7^2$. On the other hand, the difference in the parentheses is divisible by

$$10\,887 - 10\,152 = 735 = 3 \cdot 5 \cdot 7^2$$

(because the difference of the 8th powers of two numbers is divisible by the difference of the bases of the powers). It follows that N is divisible by $5 \cdot 7^2$.

2°. We have $N = (27\,195^8 - 10\,887^8) + 10\,152^8$. The number $10\,152 = 2^3 \cdot 3^3 \cdot 47$ is divisible by $2^3 \cdot 3^3$ and, on the other hand, the difference in the parentheses is divisible by

$$27\,195 - 10\,887 = 16\,308 = 2^2 \cdot 3^3 \cdot 151$$

Hence, N is divisible by $2^2 \cdot 3^3$.

Since N is divisible both by $5 \cdot 7^2$ and by $2^2 \cdot 3^3$, we conclude that N is divisible by the product of these numbers which is equal to 26 460.

62. It can easily be checked that

$$11^{10} - 1^{10} =$$
$$= (11 - 1)(11^9 + 11^8 + 11^7 + 11^6 + 11^5 + 11^4 + 11^3 + 11^2 + 11 + 1$$

The second factor on the right-hand side is obviously divisible by 10 because it is equal to a sum of 10 terms each of which ends with 1.

Thus, $11^{10} - 1$ is equal to the product of 10 by a number divisible by 10, and consequently the difference $11^{10} - 1$ is divisible by 100.

63. We have $2222^{5555} + 5555^{2222} = (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$

The number $2222^{5555} + 4^{5555}$ is divisible by $2222 + 4 = 2226 = 7 \cdot 318$ (because a sum of two odd powers is always divisible by the difference of the bases of the powers), and consequently this number is divisible by 7. The difference $5555^{2222} - 4^{2222}$ is also divisible by 7 since it is divisible by $5555 - 4 = 5551 = 7 \cdot 793$ (because the difference of any integral powers with equal exponents is divisible by the difference of the bases). As to the difference $4^{5555} - 4^{2222}$, it can be rewritten as

$$4^{2222}(4^{3333} - 1) = 4^{2222}(64^{1111} - 1)$$

whence it can readily be seen that this expression is divisible by the difference $64 - 1 = 63$ (because the difference of two integral powers with equal exponents is divisible by the difference of the bases of the powers). Consequently, $4^{5555} - 4^{2222}$ is divisible by 7.

64. We shall make use of the method of mathematical induction. A number $\overline{aaa}$ formed of three identical digits a (the bar above this expression is written in order to avoid the confusion with the product $a \cdot a \cdot a$) is divisible by 3 (because the sum of the digits of this number is equal to $3a$ and is therefore divisible by 3). Further, let us suppose that the assertion of the problem has already been proved for every number whose decimal representation consists of 3^n identical digits. We must prove that then this assertion is true for any number consisting of 3^{n+1} identical digits. Such a number can be written in the form

$$\overline{\underbrace{aa\ldots a}_{3^n \text{ times}}\;\underbrace{aa\ldots a}_{3^n \text{ times}}\;\underbrace{aa\ldots a}_{3^n \text{ times}}} = \overline{\underbrace{aa\ldots a}_{3^n \text{ times}}} \cdot 1\underbrace{00\ldots 0}_{3^n \text{ digits}}1\underbrace{00\ldots 0}_{3^n \text{ digits}}1$$

In accordance with the induction hypothesis, the first factor on the right-hand side is divisible by 3^n; the second factor is also divisible by 3 (because the sum of the digits of this factor is equal to 3). Hence, the whole product is divisible by 3^{n+1}.

65. First of all we note that the number $10^6 - 1 = 999\,999$ is divisible by 7 (because $999\,999 = 7 \cdot 142\,857$). It readily follows that the division of 10^N by 7 (where N is an arbitrary whole number) leaves the same remainder as the division by 7 of the number 10^r where r is the remainder resulting from the division of N by 6. Indeed, if $N = 6k + r$ then the number

$$10^N - 10^r = 10^{6k+r} - 10^r = 10^r(10^{6k} - 1) =$$
$$= 10^r \cdot (10^6 - 1)(10^{6k-6} + 10^{6k-12} + \ldots + 10^6 + 1)$$

is divisible by 7.

Further, the division of any integral power of 10 by 6 leaves a remainder equal to 4; indeed, according to the tests for divisibility by 2 and by 3, the difference $10^n - 4 = \underbrace{99\ldots 9}_{(n-1) \text{ times}}6$ is always divisible by $2 \cdot 3 = 6$. Thus, the remainders resulting from the division by 6 of all the exponents of the powers in the addends of the above sum are equal to 4. Consequently, when each of these 10 addends is divided by 7 we obtain the same remainder as in the case when 10^4 is divided by 7, and the division of the whole sum by 7 leaves the same remainder as the division by 7 of the number

$$10^4+10^4+10^4+10^4+10^4+10^4+10^4+10^4+10^4+10^4=$$
$$=10^5=100\,000=7\cdot 14\,285+5$$

Thus, the sought-for remainder is equal to 5.

66. (a) Every even power of 9 can be represented in the form

$$9^{2n}=81^n=\underbrace{81\cdot 81\ldots 81}_{n\text{ times}}$$

and, consequently, its last digit is 1. Every odd power of 9 can be written in the form $9^{2n+1}=9\cdot 81^n$, and therefore its last digit is 9 (because such a power is a product of a number whose last digit is 1 by the number 9). In particular, $9^{(9^9)}$ is an odd power of 9, and consequently the last digit of $9^{(9^9)}$ is equal to 9.

Now we note that any integral power of 6 ends with the digit 6; indeed, we have $6^1=6$, and if 6^n ends with 6 then the last digit of $6^{n+1}=6^n\cdot 6$ is also equal to 6. Further, the last digits of 16^n and 6^n coincide, and consequently any integral power of 16 has 6 as its last digit. Therefore any integral power of 2 with an exponent multiple of 4 ends with 6 (because $2^{4n}=16^n$). But 3^4-1 is divisible by $3+1=4$, and consequently $2^{(3^4-1)}$ ends with the digit 6 while the last digit of $2^{3^4}=2\cdot 2^{(3^4-1)}$ is 2 (because this expression is the product of a number whose last digit is 6 by 2).

(b) It is required to find the remainder resulting from the division of 2^{999} by 100 (it is clear that this remainder is formed of the last two digits of the number 2^{999}). First of all, let us show that the division of the number 2^{1000} by 25 leaves a remainder 1. Indeed, $2^{10}+1=1024+1=1025$ is divisible by 25, and consequently $2^{20}-1=(2^{10}+1)(2^{10}-1)$ is divisible by 25 while $2^{1000}-1=(2^{20})^{50}-1$ is divisible by $2^{20}-1$. It follows that the last two digits of the number 2^{1000} can be 01 or $01+25=26$ or $01+50=51$ or $01+75=76$. Since 2^{1000} is obviously divisible by 4, we see that these two digits can only be 76. Thus, 2^{999} is equal to the quotient resulting from the division by 2 of a number whose last two digits are 76, that is 2^{999} can only have the digits 38 or 88 at the end (because $76/2=38$ and $176/2=88$). Hence, since the number 2^{999} is divisible by 4 its last two digits must be 88.

Now let us find the remainder resulting from the division of the number 3^{999} by 100. We remind the reader that the last digit of every even power of 9 is 1 and that the last digit of every odd power of 9 is 9 (see the solution of Problem 65 (a)). Using these facts we can readily find the remainder resulting from the division of the number $9^5 + 1$ by 100. We have

$$9^5 + 1 = (9 + 1) \cdot (9^4 - 9^3 + 9^2 - 9 + 1) = 10 \cdot (9^4 - 9^3 + 9^2 - 9 + 1)$$

Each of the three positive summands in the algebraic sum in the parentheses ends with 1 and each of the two negative summands ends with 9. Hence, the number $9^4 + 9^2 + 1$ ends with 3 and the number $9^3 + 9$ ends with 8, and consequently the whole expression in the parentheses ends with 5. Thus, the remainder resulting from the division of the number $9^5 + 1$ by 100 is equal to $10 \cdot 5 = 50$. It follows that the number $9^{10} - 1 = (9^5+1) \cdot (9^5-1)$ is divisible by 100 and, since $3^{1000} - 1 = 9^{500} - 1 = (9^{10})^{50} - 1$ is divisible by $9^{10} - 1$ (because a difference of two integral powers is divisible by the difference of the bases of the powers), the number $3^{1000} - 1$ is also divisible by 100. Therefore the number 3^{1000} ends with the digits 01. Further, 3^{1000} is divisible by 3, and consequently if the integral number of hundreds contained in 3^{1000} is divided by 3 we must obtain 2 in the remainder (if the division of this number of hundreds by 3 gave 1 or 0 in the remainder then the number of hundreds plus 01 could not be divisible by 3). We see that the last two digits of the number $3^{999} = 3^{1000} . 3$ must be the same as those of the number $201/3 = 67$.

(c) We have to find the remainder obtained when the number $14^{(14^{14})} = (7 \cdot 2)^{(14^{14})}$ is divided by 100 because this remainder consists of the last two digits of the number $14^{(14^{14})}$. To this end we shall separately determine the remainders resulting from the division of the numbers $7^{(14^{14})}$ and $2^{(14^{14})}$ by 100.

The number $7^4 - 1 = 2401 - 1 = 2400$ is divisible by 100. It follows that if $n=4k$ (i.e. if n is divisible by 4) then $7^n - 1$ is divisible by 100 (because $7^{4k} - 1 = (7^4)^k - (1)^k$ is divisible by $7^4 - 1$). Further, $14^{14} = 2^{14} \cdot 7^{14}$ is divisible by 4, and consequently $7^{(14^{14})} - 1$ is divisible by 100, whence it follows that the last two digits of the number $7^{(14^{14})}$ are 01.

As was shown in the solution of Problem 65 (b), $2^{20} - 1$ is divisible by 25; consequently, if $n = 20k$ (i.e. if n is divisible by 20) then $2^n - 1$ is divisible by 25. Now let us find the remainder resulting from the division of the number 14^{14} by 20. We obviously have $14^{14} = 2^{14} \cdot 7^{14}$. Further, we have $2^{14} = 4 \cdot 2^{12}$, and since $2^{12} - 1 = (2^4)^3 - 1$ is divisible by $2^4 - 1 = 16 - 1 = 15$, we see that $4(2^{12} - 1)$ is divisible by 20; consequently when $2^{14} = 4 \cdot 2^{12}$ is divided by 20 we obtain 4 in the remainder. Besides, we have $7^{14} = 49 \cdot 7^{12}$, and since the division of 7^{12} by 20 leaves a remainder of 1 (because 12 is divisible by 4 and therefore $7^{12} - 1$ is divisible by 100), we see that the remainders resulting from the division of $49 \cdot 7^{12}$ by 20 and from the division of 49 by 20 coincide and are equal to 9. Thus, the division of $14^{14} = 2^{14} \cdot 7^{14}$ by 20 leaves the same remainder as the division of the product $4 \cdot 9 = 36$ by 20, that is this remainder is equal to 16 because $14^{14} = 20K + 16$. Now it readily follows that the division by 25 of the numbers $2^{(14^{14})} = 2^{16} \cdot 2^{20K}$ and $2^{16} = 65\,536$ leaves the same remainders, which means that the last two digits of the number $2^{(14^{14})}$ can only be 11; 36; 61 or 86. Since the number $2^{(14^{14})}$ is divisible by 4 we conclude that this number has the digits 36 at the end.

Thus the last two digits of the number $7^{(14^{14})}$ are 01 and those of the number $2^{(14^{14})}$ are 36. Consequently, their product $7^{(14^{14})} \cdot 2^{(14^{14})} = 14^{(14^{14})}$ ends with the digits 36.

67. (a) It is clear that both numbers 9^{9^9} and $9^{9^{9^9}}$ end with 9 (cf. the solution of Problem 66 (a)), that is their last digits coincide. Further, by Newton's binomial formula, we have

$$A = 9^{9^9} = (10 - 1)^{9^9} =$$
$$= 10^a - C(a, 1) \cdot 10^{a-1} + C(a, 2) \cdot 10^{a-2} - \dots + C(a, 1) \cdot 10 - 1$$

and

$$B = 9^{9^{9^9}} = (10 - 1)^{9^{9^9}} =$$
$$= 10^b - C(b, 1) \cdot 10^{b-1} + C(b, 2) \cdot 10^{b-2} - \dots + C(b, 1) \cdot 10 - 1$$

where $a = 9^9$ and $b = 9^{9^9}$. Thus, the last two digits of the numbers under consideration coincide with the last two digits of the numbers

$$C(a, 1) \cdot 10 - 1 = 10a - 1 \quad \text{and} \quad C(b, 1) \cdot 10 - 1 = 10b - 1$$

respectively. Further, both numbers $a = 9^9$ and $b = 9^{9^9}$ end with the digit 9 (see again the solution of Problem 66 (a)). Therefore the last two digits of the numbers $10a$ and $10b$ are 90 and those of the numbers A and B are 89.

(b) By analogy with the solution of Problem 67 (a), we can find the last six digits of the two numbers in question (the solution of Problem 67 (a) is based on the equality $9 = 10 - 1$; in the present problem the role of this equality is played by the relation $7^2 = 50 - 1$). It turns out that these digits coincide. However, this method of the solution leads to rather lengthy calculations (because here, instead of two last digits, we deal with six last digits) and therefore it is preferable to use another method.

We must show that the last six digits of the numbers $A=7^a$ and $B=7^b$ coincide where $a=7^{7^{7^7}}$ and $b=7^{7^7}$. Hence, we must prove the difference

$$A-B=7^a-7^b=7^b\left(7^{a-b}-1\right)$$

is divisible by $1\,000\,000=2^6\cdot 5^6$ (this means that we must prove that the number

$$D=7^d-1$$

where $d=a-b$ is divisible by $1\,000\,000=2^6\cdot 5^6$). Thus, the problem reduces to the *determination of the exponents of the power of two and of the power of five by which the number* $D=7^d-1$ *is divisible* where d is a natural number. We shall investigate separately the divisibility of the number D by 2^α (in this case we have to prove that $\alpha\geqslant 6$) and the divisibility of D by 5^β.

1°. Let $C=7^c-1$ where $c=2^pq$ (here q is odd). We shall prove that in this case we have $C=2^{\alpha_p}P$ where P is an odd number and the exponent α_p (which is dependent on p solely and does not depend on q) satisfies the recurrence relation

$$\alpha_p=\alpha_{p-1}+1 \qquad (*)$$

for $p>1$.

Since the number $7^{(2^1)}-1=7^2-1=48$ is divisible by $2^4=$ $=16$ and is not divisible by 2^5 we see that $\alpha_1=4$, and therefore (*) obviously implies that

$$\alpha_p=p+3 \quad \text{for all} \quad p\geqslant 1$$

Hence, *if the greatest exponent of the power of two by which the number c is divisible is equal to $p\geqslant 1$ then the greatest exponent α_p of the power of two by which the number $C=7^c-1$ is divisible is equal to $p+3$* (for $p=0$ we have $\alpha_0=1$ because the number $7^{2^0}-1=7-1=6$ is divisible by 2 and is not divisible by any higher power of 2).

We shall first prove that the number α_p is independent of q. This follows from the formula

$$7^{(2^pq)}-1=\left(7^{2^p}\right)^q-1^q=$$
$$=\left[7^{2^p}-1\right]\left\{\left[7^{(2^p)}\right]^{q-1}+\left[7^{(2^p)}\right]^{q-2}+\left[7^{(2^p)}\right]^{q-3}+\ldots+1\right\}$$

where the sum in the curly brackets consisting of the odd number q of odd summands is obviously odd. This implies that *the highest powers of 2 by which the numbers $7^{2^pq}-1$ and $7^{2^p}-1$ are divisible coincide.* Therefore in the further argument we can put $q=1$, that is we can replace the number $C=7^{2^pq}-1$ by the number $C_p = 7^{2^p} - 1$.

Now it only remains to make use of the formula

$$C_p = 7^{2^p} - 1 = (7^{2^{p-1}})^2 - 1^2 = [7^{2^{p-1}} - 1][7^{2^{p-1}} + 1] = C_{p-1} \cdot C'$$

The division of 7 by 4 leaves a remainder equal to -1, and therefore when 7^m is divided by 4 we obtain -1 in the remainder for odd m and $+1$ for even m. Consequently, *the division of* $7^{2^n}+1$ (and even of $7^{2n}+1$) *by* 4 *leaves a remainder equal to* 2 *for any* (natural) n. This means that $7^{2^n}-1$ *and* $7^{2n}-1$ *are divisible by* 2 *but are not divisible by* 2^2. Hence, the left-hand member C_p of the equality $C_p = C_{p-1} \cdot C'$ is divisible by the number 2^{α_p}, C_{p-1} is divisible by $2^{\alpha_{p-1}}$ and C' is divisible by 2 and not divisible by a higher power of 2, which implies formula (*). (It is clear that for $p = 1$ we have a special case because $C' = 7^{2^0} + 1 = 7 + 1 = 8$ is divisible not only by 2^1 but also by 2^3; the distinction appears because this is the only case when the exponent 2^0 of the power of 7 in the expression of C' is an odd number.)

Thus, in order to determine the exponent α in the formula $D = 7^d - 1 = 2^\alpha \cdot Q$ (where Q is odd) we should only find by what power of the number $d = a - b = 7^{a_1} - 7^{b_1} = 7^{b_1}(7^{d_1} - 1)$ is divisible (that is by what power of two the number $7^{d_1} - 1$ is divisible) where $a_1 = 7^{7^{7^7}}$, $b_1 = 7^7$ and $ad_1 = a_1 - b_1$. As we know, to this end it is necessary and sufficient to determine the power of two by which the exponent $d_1 = a_1 - b_1 = 7^{a_2} - 7^{b_2} = 7^{b_2}(7^{d_2} - 1)$ is divisible where $a_2 = 7^{7^7}$, $b_2 = 7$ and $d_2 = a_2 - b_2 = 7^{a_3} - 7^{b_3} = 7^{b_3}(7^{d_3} - 1)$ (here $b_3 = 1$, $a_3 = 7^7$ and $d_3 = a_3 - b_3 = 7^7 - 1$). Since 7 is an odd number, the number $d_3 = 7^7 - 1$ is divisible only by $2^{\alpha_0} = 2^1 = 2$. It follows that the number $7^{d_3} - 1$ is divisible by $2^{\alpha_1} = 2^4$; thus, the number $d_2 = 7^{b_3}(7^{d_3} - 1)$ is divisible by 2^4 and, consequently, the number $7^{d_2} - 1$ is divisible by $2^{\alpha_4} = 2^7$. Now, since $d_1 = 7^{b_2}(7^{d_2} - 1)$ is divisible by 2^7, the number $7^{d_1} - 1$ is divisible by $2^{\alpha_7} = 2^{10}$, and therefore we conclude that $d = 7^{b_1}(7^{d_1} - 1)$ is also divisible by 2^{10}. Hence, $D = 2^d - 1$ is divisible by $2^{\alpha_{10}} = 2^{13}$.

2°. The divisibility of the number $C = 7^c - 1$ (where it is advisable to put $c = 5^r s$) by the powers of five can be investigated in a completely analogous manner. In this case we should however stipulate that the number s (which is not divisible by 5) is divisible by 2^2 (or by a higher power of 2) because if s is odd or is divisible by 2 but not divisible by 4 we arrive at some other conclusions. For $s = 4s_1$, that is for $c = 5^r \cdot 4s_1$ where s_1 is an integral number, the greatest exponent β_r of the power of five by which the number C is divisible is dependent solely on the exponent r in the formula for c, that is the number β_r is independent of s_1; as before, there holds the recurrence relation

$$\beta_r = \beta_{r-1} + 1 \qquad (**)$$

which is analogous to (*). Since the number $7^4 - 1 = (7^2)^2 - 1 = (7^2 + 1)(7^2 - 1) = 50 \cdot 48$ is divisible by $5^2 = 25$, that is $\beta_0 = 2$, relation (**) implies that *for all $r \geqslant 0$ we have*

$$\beta_r = r + 2$$

Hence, *if the exponent c in the expression $C = 7^c - 1$ is divisible by 4, and if the greatest exponent of the power of five by which c is divisible is equal to r then the greatest exponent β_r of the power of five by which C is divisible is equal to $r + 2$.*

To prove the independence of β_r of s_1 it suffices to make use of the formula

$$C = 7^c - 1 = 7^{5^r \cdot 4s_1} - 1 = \left(7^{5^r \cdot 4}\right)^{s_1} - 1^{s_1} =$$
$$= [7^{5^r \cdot 4} - 1]\left\{[7^{(5^r \cdot 4)}]^{s_1 - 1} + [7^{(5^r \cdot 4)}]^{s_1 - 2} + \ldots + 7^{(5^r \cdot 4)} + 1\right\}$$

Since the division of $7^4 = (7^2)^2 = (50 - 1)^2 = 50^2 - 2 \cdot 50 + 1$ by 5 leaves a remainder equal to 1, the division of the number $(7^4)^n$ by 5 also leaves a remainder 1 for any n. Therefore the expression in the curly brackets on the right-hand side of the last formula is a sum of s_1 numbers the division of each of which by 5 leaves a remainder equal to 1, whence it follows that this sum is not divisible by 5 because s_1 is not divisible by 5. Therefore *the number $C = 7^{5^r \cdot 4s_1} - 1$ is divisible by the same power of five by which the number $E_r = 7^{5^r \cdot 4} - 1$ is divisible.* This allows us to put $s_1 = 1$ in the further course of the argument, that is we can replace the number C by the number E_r.

Further, we have

$$E_r = 7^{5^r \cdot 4} - 1 = \left(7^{5^{r-1} \cdot 4}\right)^5 - 1^5 =$$
$$= \left(7^{5^{r-1} \cdot 4} - 1\right)\left\{\left(7^{5^{r-1} \cdot 4}\right)^4 + \left(7^{5^{r-1} \cdot 4}\right)^3 + \right.$$
$$\left. + \left(7^{5^{r-1} \cdot 4}\right)^2 + 7^{5^{r-1} \cdot 4} + 1\right\} = E_{r-1} \cdot E'$$

First of all, this implies that E_r is divisible by E_{r-1}, that is the exponent of the power of five by which E_r is divisible is not less than the exponent of the power of five by which E_{r-1} is divisible. In other words, we have $\beta_r \geqslant \beta_{r-1}$; since $\beta_0 = 2$, it follows that there hold the inequalities

$$2 = \beta_0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \ldots$$

Thus, the number $E_{r-1} = 7^{5^{r-1}\cdot 4} - 1$ is divisible by 25 for all $r \geqslant 1$, and consequently the remainder resulting from the division of the number $e_{r-1} = e = 7^{5^{r-1}\cdot 4}$ by 25 is equal to 1, and the division of any power e^k of the number e by 25 also leaves a remainder equal to 1. Since the expression E' in the curly brackets on the right-hand side of the formula for E_r is equal to the sum $e^4 + e^3 + e^2 + e + 1$, the division of E' by 25 leaves a remainder equal to 5, that is E' is divisible by 5^1 and is not divisible by 5^2. What has been established and the formula $E_r = E_{r-1} \cdot E'$ imply relation (**).

Let us come back to the number $A - B = 7^b \cdot D$ where $D = 7^d - 1$. As has been shown, the number d is divisible by 4 (it is even divisible by $2^{10} = 1024$), and therefore it only remains to determine the power of 5 by which the number $d = 7^{a_1} - 7^{b_1} = 7^{b_1}(7^{d_1} - 1)$ is divisible where $d_1 = a_1 - b_1$ (see the end of Section 1° of the solution of the present problem). Further, since d_1 is also divisible by 4 (this number is even divisible by 2^7), the problem reduces to the determination of the exponent of the power of five by which the number $d_1 = 7^{a_2} - 7^{b_2} = 7^{b_2}(7^{d_2} - 1)$ is divisible where the number $d_2 = a_2 - b_2 = 7^{a_3} - 7^{b_3} = 7^{b_3}(7^{d_3} - 1)$ is divisible by 4 (it is even divisible by 2^4) and $d_3 = 7^7 - 1$. As we know, the number d_3 is divisible by 2 and is not divisible by 4, that is $d_3 = 2f$ where f is *odd*; therefore the number $7^{d_3} - 1 = 49^f - 1 = (50-1)^f - 1$ is not divisible by 5 (its division by 5 leaves a remainder of -2 or, which is the same, a remainder equal to 3). On the other hand, the number $7^{d_3} - 1$ and also the number $d_2 = 7^{b_3}(7^{d_3} - 1)$ are divisible by 4, whence it follows that the number $7^{d_2} - 1$ is divisible by $5^{\beta_0} = 5^2$ and is not divisible by any higher power of five; therefore $d_1 = 7^{b_2}(7^{d_2} - 1)$ is divisible by 5^2 and $d = 7^{b_1}(7^{d_1} - 1)$ is divisible by $5^{\beta_2} = 5^4$ while the expression $D = 7^d - 1$ and the number $A - B = 7^b \cdot D$ we are interested in are divisible by $5^{\beta_4} = 5^6$.

This argument concludes the solution of the problem.

Remark. It is clear that, by a complete analogy with the solution of this problem, it can be shown that the numbers A_{n+2} and A_n composed of $n+2$ and n digits 7 respectively (A_{n+2} and A_n are similar to the numbers B and A considered in the present problem) have $2n-2$ identical digits at the end of their decimal representations. We recommend the reader to try to estimate the number of identical digits at the end of the numbers A_n and A_m composed of n and of m digits 7 respectively (here it is natural to begin with the case when the difference $n - m$ is not very large).

68. **(a)** When two numbers one of which ends with a digit a while the other ends with a digit b are multiplied by each other the last digit of their product coincides with that of the product ab This proposition allows us to solve the given problem rather simply. Let us perform consecutively the raisings to the power and consider only the last digits of the resulting numbers: the last digit of 7^2 is 9, the last digit of $7^3 = 7^2 \cdot 7$ is 3, the last digit of $7^4 = 7^3 \cdot 7$ is 1 and the last digit of $7^7 = 7^4 \cdot 7^3$ is 3.

Further, in just the same way we find that the last digit of $(7^7)^7$ is again equal to 7 (indeed, $(7^7)^2$ ends with the digit 9, $(7^7)^3$ ends with the digit 7, $(7^7)^4$ ends with the digit 1 and $(7^7)^7$ ends with the digit 7). It follows that the number $((7^7)^7)^7$ has the same last digit as the number 7^7, that is this last digit is equal to 3, and the last digit of the number $(((7^7)^7)^7)^7$ is again equal to 7 etc. Continuing the argument in the same manner we conclude that after an odd number of raisings to the 7th power we every time obtain a number with the last digit 3 and after an even number of raisings to the 7th power we obtain a number with the last digit 7. Since the number 1000 is even, the number we are interested in ends with the digit 7.

Now let us consider two numbers whose last two digits form two 2-digit numbers A and B respectively. It is evident that the product of the given numbers has the same last *two digits* as the product $A \cdot B$. This allows us to determine the last two digits of the number we are interested in. As before, we check that the last two digits of 7^7 are 43 and that the last two digits of $(7^7)^7$ coincide with those of 43^7, namely $(7^7)^7$ ends with 07. It follows that if we consecutively raise the numbers 7, 7^7, $(7^7)^7$, ... to the 7th power then after an odd number of these operations we every time arrive at a number whose last two digits are 43 and after an even number of the operations we arrive at a number ending with the digits 07. Consequently, the last two digits of the sought-for number are 07.

(b) As was shown in the solution of Problem 68 (a), the number 7^4 ends with the digit 1. It follows that the last digit of $7^{4k} = (7^4)^k$ is also equal to 1 and that 7^{4k+l} where l is one of the numbers 0, 1, 2 or 3 has the same last digit as 7^l ($7^{4k+l} = 7^{4k} \cdot 7^l$). Hence, the problem reduces to the determination of the remainder resulting from the division by 4 of the exponent of the power to which 7 should be raised in order to get the number mentioned in the condition of the problem.

The exponent of the power to which the number 7 is raised in this problem is itself a power of 7 with a very large exponent. We have to determine the remainder which is obtained when the latter power of seven is divided by 4. Since $7 = 8 - 1$, it follows that the remainder resulting from the division of $7^2 = (8-1) \cdot (8-1)$ by 4 is equal to 1, the division of $7^3 = 7^2 \cdot (8-1)$ by 4 leaves a remainder equal to -1 or, which is the same, equal to 3, and, generally, the division of any even power of 7 by 4 leaves a remainder of 1 and the division of any odd power of 7 by 4 leaves a remainder equal to -1 or, which is the same, equal to $+3$. Further, the exponent of the power of 7 we consider in this problem is an odd number because it is itself a power of 7. Consequently, the number mentioned in the condition of the problem is of the form 7^{4k+3} and hence its last digit coincides with that of 7^3, that is this digit is equal to 3.

Since 7^4 ends with the digits 01 we conclude that the last *two* digits of 7^{4k+l} coincide with those of 7^l. Consequently, the number in question ends with the same two digits as the number 7^3, that is these digits are 43.

69. Let us consider, in succession, the following numbers:

1°. $Z_1 = 9$

2°. $Z_2 = 9^{Z_1} = (10-1)^{Z_1} = 10^{Z_1} - C(Z_1,\ 1)\cdot 10^{Z_1-1} + \ldots$ $\ldots + C(Z_1,\ 1)\cdot 10 - 1$

where the terms designated by the dots are all divisible by 100. Since $C(Z_1,\ 1) = 9$, the last two digits of the number Z_2 coincide with those of the number $9\cdot 10 - 1 = 89$.

3°. $Z_3 = 9^{Z_2} = (10-1)^{Z_2} = 10^{Z_2} - C(Z_2,\ 1)\cdot 10^{Z_2-1} + \ldots$ $\ldots - C(Z_2,\ 2)\cdot 10^2 + C(Z_2,\ 1)\cdot 10 - 1$

The number Z_2 has 89 at the end; consequently, the last two digits of $C(Z_2,\ 1) = Z_2$ are 89 and the last digit of $C(Z_2,\ 2) = \frac{Z_2(Z_2-1)}{1\cdot 2} = \frac{\ldots 89\cdot\ldots 88}{1\cdot 2}$ (where the dots designate the unknown digits) is 6. Consequently, the last three digits of the number Z_3 coincide with the last three digits of the number $-600 + 890 - 1 = 289$.

4°. $Z_4 = 9^{Z_3} = (10-1)^{Z_3} = 10^{Z_3} - C(Z_3,\ 1)\cdot 10^{Z_3-1} + \ldots$ $\ldots + C(Z_3,\ 3)\cdot 10^3 - C(Z_3,\ 2)\cdot 10^2 + C(Z_3,\ 1)\cdot 10 - 1$

Since Z_3 ends with 289, the last three digits of $C(Z_3,\ 1) = Z_3$ are 289. The number

$$C(Z_3,\ 2) = \frac{Z_3(Z_3-1)}{1\cdot 2} = \frac{\ldots 289\cdot\ \ldots 288}{1\cdot 2}$$

ends with 16. The last digit of the number

$$C(Z_3,\ 3) = \frac{Z_3(Z_3-1)(Z_3-2)}{1\cdot 2\cdot 3} = \frac{\ldots 289\cdot\ \ldots 288\cdot\ \ldots 287}{1\cdot 2\cdot 3}$$

is equal to 4. Consequently, the last four digits of the number Z_4 coincide with those of the number $4000 - 1600 + 2890 - 1 = 5289$.

5°. $Z_5 = 9^{Z_4} = (10-1)^{Z_4} = 10^{Z_4} - C(Z_4,\ 1)\cdot 10^{Z_4-1} + \ldots$ $\ldots - C(Z_4,\ 4)\cdot 10^4 + C(Z_4,\ 3)\cdot 10^3 - C(Z_4,\ 2)\cdot 10^2 + C(Z_4,\ 1)\cdot 10 - 1$

Since Z_4 ends with 5289, the last four digits of $C(Z_4, 1) = Z_4$ are 5289. The number

$$C(Z_4, 2) = \frac{Z_4(Z_4-1)}{1\cdot 2} = \frac{\dots 5289\cdot \dots 5288}{1\cdot 2}$$

ends with 116.

The number

$$C(Z_4, 3) = \frac{Z_4(Z_4-1)(Z_4-2)}{1\cdot 2\cdot 3} = \frac{\dots 5289\cdot \dots 5288\cdot \dots 5287}{1\cdot 2\cdot 3}$$

ends with 64, and, finally, the last digit of the number

$$C(Z_4, 4) = \frac{Z_4(Z_4-1)(Z_4-2)(Z_4-3)}{1\cdot 2\cdot 3\cdot 4} =$$

$$= \frac{\dots 5289\cdot \dots 5288\cdot \dots 5287\cdot \dots 5286}{1\cdot 2\cdot 3\cdot 4}$$

is equal to 6. Hence, the last five digits of Z_5 coincide with the five digits of the number

$$-60\,000 + 64\,000 - 11\,600 + 52\,890 - 1 = 45\,289$$

Further, the coincidence of the last four digits of the number Z_5 with the last four digits of the number Z_4 implies that the last five digits of the number $Z_6 = 9^{Z_5} = (10-1)^{Z_5}$ coincide with the last five digits of the number $Z_5 = 9^{Z_4}$. In the same way we can show that all the numbers belonging to the sequence

$$Z_5, \quad Z_6 = 9^{Z_5}, \quad Z_7 = 9^{Z_6}, \dots, Z_{1000} = 9^{Z_{999}}, \quad Z_{1001} = 9^{Z_{1000}}$$

end with the same five digits, namely with 45 289. The number Z_{1001} is nothing other than the number N mentioned in the condition of the problem.

70. First of all let us find the remainders resulting from the division of the numbers 5^n and n^5 by 13 for several consecutive values $n = 0, 1, 2, \dots$. It is more convenient to begin with the numbers 5^n; we can write the following table of the remainders:

n	0	1	2	3	4	...
The number 5^n	1	5	25	125	625	...
The remainder resulting from the division of 5^n by 13	1	5	−1	−5	1	...

(Here we write the remainder −1 instead of the remainder 12 and the remainder −5 instead of the remainder 8; this facilitates the determination of all the other remainders: if the division of 5^n by 13 leaves a remainder equal to −1, that is if $5^n = 13k - 1$ where

k is an integer, then the division of $5^{n+1} = 5^n \cdot 5 = (13k - 1)5 = 13(5k) - 5$ by 13 leaves a remainder of -5. Similarly, if the division of 5^m by 13 leaves a remainder equal to -5, that is if $5^m = 13l - 5$, then the remainder resulting from the division of $5^{m+1} = 5^m \cdot 5 = (13l - 5) \cdot 5 = 13(5l) - 25 = 13(5l - 2) + 1$ by 13 is equal to 1.) There is no need to continue this table of the remainders because, since $5^4 = 13q + 1$, the division of $5^5 = 5^4 \cdot 5 = (13q + 1)5$ by 13 leaves the same remainder as the division of 5 by 13, that is the remainder equal to 5; similarly, the division of the number $5^6 = 5^4 \cdot 5^2 = (13q + 1) \cdot 5^2$ by 13 leaves the same remainder as the division of the number 5^2 by 13 (that is the remainder equal to -1), etc. Thus, *in this sequence of the remainders the numbers* 1, 5, -1 *and* -5 *alternate in succession.*

We can similarly compile the table of the remainders obtained when the numbers n^5 are divided by 13 where $n = 0, 1, 2, \ldots,$ etc. The division of the number

$$(13p + r)^5 = \underbrace{(13p + r)(13p + r) \ldots (13p + r)}_{5 \text{ factors}}$$

by 13 leaves the same remainder as the division of the number r^5, and therefore we can limit ourselves to the values $n = 0, 1, 2, 3, \ldots, 12$. If the number n is equal to s or if its division by 13 leaves a remainder equal to s and if the division of the number n^2 by 13 leaves a remainder t then the division of the numbers $n^5 = n^2 \cdot n^2 \cdot n$ and $t \cdot t \cdot s$ by 13 leaves one and the same remainder. This facilitates the compiling of the required table for the values of n equal to 4, 5, and 6. Finally, it should be noted that if the division of the number n^5 by 13 leaves a remainder u then the remainder resulting from the division of the number $(13 - n)^5 = \underbrace{(13 - n)(13 - n) \ldots (13 - n)}_{5 \text{ factors}}$ by 13 coincides with the remainder resulting from the division of the number $(-n)^5$ by 13, this remainder being equal to $-u$ or, equivalently, to $13 - u$.

Now we can write down the corresponding table of the remainders:

n	0	1	2	3	4	5
n^5	0	1	32	243		
n^2					16	25
The remainder resulting from the division of n^2 by 13					3	-1
The remainder resulting from the division of n^5 by 13	0	1	6	-4	-3 (because $t \cdot t \cdot s = 3 \cdot 3 \cdot 4$)	$-1(-1)5=5$

n	6	7	8	9	10	11	12	. . .
n^5								
n^2	36							
The remainder resulting from the division of n^2 by 13	—3							
The remainder resulting from the division of n^5 by 13	2 (because $t \cdot t \cdot s =$ $(-3) \cdot (-3) \cdot 6$)	—2	—5	3	4	—6	—1	. . .

Here, when writing the remainders corresponding to the values of n equal to 7, 8, ..., 12, we take into account that the division of the numbers n^5 and $(13-n)^5$ by 13 leaves remainders equal to u and $-u$ respectively. Besides, for the values of n exceeding 12 the same remainders 0, 1, 6, —4, —3, 5, 2, —2, —5, 3, 4, —6 and —1 repeat periodically in the table.

We see that the first table has 4 numbers in the "period of the remainders" while the second table has 13 numbers in the "period of the remainders"; therefore the number $4 \cdot 13 = 52$ determines the "length of the period of the remainders" in the "union" of both tables in the sense that when n is increased by 52 (or by any number multiple of 52) the remainders resulting from the division of the numbers 5^n and n^5 by 13 do not change. It is clear that we can limit ourselves to the consideration of only those columns of the second table which correspond to the remainders ± 1 and ± 5 because in the first table only the remainders 1, 5, —1 and —5 alternate. Further, in the second table for the values of n ranging from 0 to 51 the remainders 1 correspond to the values of n equal to 1, $1+13=14$, $1+2 \cdot 13=27$ and $1+3 \cdot 13=40$. Among these four numbers 1, 4, 27 and 40 only the number 14 is of the form $4x+2$, and in the first table to the number $n=14$ there corresponds a remainder equal to —1. Thus, the number $n=14$ satisfies the required condition because $5^{14}+14^5$ is divisible by 13. Similarly, in the second table, for the same values of n ranging from 0 to 51, the remainders —1 correspond to the values of n equal to 12, $12+13=25$, $12+2 \cdot 13=38$ and $12+3 \cdot 13=51$; among these four numbers 12, 25, 38 and 51 only 12 has the form $4y$, the remainder corresponding to $n=12$ in the first table being equal to 1. Similarly, in the second table the remainders 5 correspond to the values of n equal to 5, $5+13=18$, $5+2 \cdot 13=31$ and $5+3 \cdot 13=44$, and the remainders —5 correspond to the values of n equal to 8, $8+13=21$, $8+2 \cdot 13=34$ and $8+$ $+3 \cdot 13=47$; further, among the four numbers 5, 18, 31 and 44 only 31 is of the form $4z+3$ for which the division of $5^{4z+3}=5^{31}$ by 13 leaves a remainder equal to —5 while among the four numbers 8, 21, 34 and 47 only 21 has the form $4w+1$ for which the division $5^{4w+1} = 5^{21}$ by 13 leaves a remainder equal to 5. Thus, within the limits from $n = 0$ to $n = 52$ (that is for $0 \leqslant n \leqslant 52$) only the four natural numbers $n = 12, 14, 21$ and 31 satisfy the required condition. As to the whole set of the natural numbers satisfying the condition of the problem, it consists of the following four sequences:

$$n = 52m + 12, \quad n = 52m + 14 \quad (\text{that is } n = 26(2m) + 12 \text{ and } n = 26(2m+1) - 12),$$

$$n = 52m + 21 \text{ and } n = 52m + 31 \quad (\text{that is } n = 52m \pm 21)$$

where $m = 0, 1, 2, \ldots$ (the only exception is that in the formula $n = 52m - 21$ we should put $m > 0$).

Now it becomes clear that the *smallest* number n satisfying the conditions of the problem is $n = 12$.

71. It is clear that the last two digits of the numbers $n^2, n^3, \ldots$ where n is a nonnegative integer depend solely on the last two digits of the number n, which follows from the ordinary arithmetic rule for the multiplication of multiplace numbers written as a column. On the other hand, the last two digits of 100 consecutive nonnegative integers must necessarily run over the sequence 00, 01, 02, ..., 99 (although, in the general case, their order may differ from the one in which we have written the sequence here). Therefore the problem reduces to the determination of the last two digits of the sum

$$N_a = 0^a + 1^a + 2^a + \ldots + 99^a$$

for $a = 4$ and $a = 8$.

(a) If $n = 10x + y$ is a two-digit number then

$$n^4 = (10x + y)^4 = 10^4x^4 + 4 \cdot 10^3x^3y + 6 \cdot 10^2x^2y^2 + 4 \cdot 10xy^3 + y^4$$

and the last two digits of the number n^4 depend solely on the last two terms of this sum because each of the other terms has two noughts at the end. Consequently, it only remains to determine the last two digits of the numbers equal to the sums

$$\sum_x \sum_y 4 \cdot 10xy^3 \quad \text{and} \quad \sum_x \sum_y x^0y^4 = 10 \sum_y y^4$$

where x and y independently run over the values ranging from 0 to 9 (here we put $0^0 = 1$, and therefore $\sum\limits_x x^0 = 0^0 + 1^0 + \ldots + 9^0 = 10$).

Now we note for any fixed y the number

$$\sum_x 4\cdot 10xy^3 =$$
$$= 4\cdot 10\,(0+1+2+3+4+5+6+7+8+9)\cdot y^3 = 1800y^3$$

ends with two noughts; therefore the number $\sum_x \sum_y 4\cdot 10xy^3$ also has two noughts at the end, and hence it does not affect the last two digits of the number N_4. Thus, we have to determine the last two digits of the number

$$10\sum_y y^4 =$$
$$= 10\,(0^4+1^4+2^4+3^4+4^4+5^4+6^4+7^4+8^4+9^4) = 10S_4$$

The last equality shows that it suffices to find the last digit of the number equal to the sum S_4.

To determine the last digit of S_4 let us consider the following table in which the last digits of the numbers y, y^2 and y^4 are written in succession:

y	0	1	2	3	4	5	6	7	8	9
y^2	0	1	4	9	6	5	6	9	4	1
y^4	0	1	6	1	6	5	6	1	6	1

It follows that the last digit of the number S_4 coincides with that of the sum

$$0+1+6+1+\ldots+1 = 4(1+6)+5 = 33$$

and consequently the last two digits of the number N_4 and also of the number mentioned in the condition of the problem are 30.

(b) By complete analogy with the solution of Problem 71 (a), we find that for $a=8$ the last two digits of the sum N_8 (and, consequently, the last two digits of the number we are interested in) coincide with the last two digits of the number $10\,S_8$ where $S_8 = 0^8+1^8+\ldots+9^8$. From the table of the last two digits of the number y^4 written above and from the equality $y^{4k}=(y^4)^k$ it follows that *the number* y^{4k} (where y is a digit) *has the same last digit as the number* y^4. Therefore in the case when $a=8$ the last two digits of the number in question are the same as in the case when $a=4$, that is these digits are 30.

Remark. It can easily be seen that the same result can be obtained for all values of a *multiple* of 4, that is for $a = 4, 8, 12, 16, \ldots$.

72. According to the formula for the sum of the members of a geometric progression, we have

$$N = \frac{50^{1000}-1}{50-1} = \frac{50^{1000}-1}{49}$$

The number 1/49 is changed into a repeating decimal whose period consists of 42 digits and can simply be found by division:

$$\frac{1}{49}=0.(020408163265306122448979591836734693877551)$$

(here the parentheses symbolize the period of the decimal). In the abbreviated form we can write

$$\frac{1}{49}=0.(P)$$

where the symbol P designates the above sequence of 42 digits and (P) designates the period.

The nearest integer to 1000 multiple of 42 is equal to $1008 = 24\cdot 42$. Hence,

$$\frac{10^{1008}}{49}=10^{1008}\cdot\frac{1}{49}=\underbrace{PP\ldots P}_{24\text{ times}}\mathbf{.}P\ldots$$

where the heavy face type dot between two neighbouring P's symbolizes the decimal point.

Thus, the fraction $M=(10^{1008}-1)/49$ can be written as

$$M=\frac{10^{1008}-1}{49}=10^{1008}\cdot\frac{1}{49}-\frac{1}{49}=\underbrace{PP\ldots P}_{24\text{ times}}$$

and therefore it is equal to a whole number written with the aid of 1008 digits which can be divided into 24 repeating groups of the 42 digits denoted as P (by the way, M is in fact a 1007-digit whole number because the sequence of digits denoted as P begins with nought).

Now let us form the difference between the number N we are interested in and the number M:

$$N-M=\frac{5^{1000}\cdot 10^{1000}-1}{49}-\frac{10^{1008}-1}{49}=\frac{5^{1000}-10^{8}}{49}\cdot 10^{1000}$$

Since the difference $N-M$ of two integral numbers is itself an integral number and since 10^{1000} is relatively prime to 49, the number $5^{1000}-10^{8}$ must be divisible by 49. Consequently, $x=(5^{1000}-10^{8})/49$ is an integral number and the difference $N-M=10^{1000}\cdot x$ ends with 1000 noughts. Thus, the last 1000 digits of the number N coincide with those of the number M, namely they form the sequence

$$p\underbrace{PP\ldots P}_{23\text{ times}}$$

where p is a group of 34 digits which are the last 34 digits of the number P.

73. Let $M = 10A + a$ denote the original number where a is the last digit of M. Then the number N obtained from M as described in the condition of the problem is obviously equal to $a \cdot 10^{6n-1} + A$ where $6n$ is the number of the digits forming the number M.

Let us consider the expression

$$M - 3N = (10A + a) - (3 \cdot 10^{6n-1}a + 3A) =$$
$$= 7A - (3 \cdot 10^{6n-1} - 1)\, a$$

According to the condition of the problem, the minuend M on the left-hand side is divisible by 7; the number $7A$ is obviously divisible by 7; therefore if we prove that the number $3 \cdot 10^{6n-1} - 1$ and, consequently, the number $(3 \cdot 10^{6n-1} - 1)a$, are divisible by 7, this will imply that the number $3N$ and, consequently, the number N, are also divisible by 7.

The division of the number 10 by 7 leaves a remainder 3 and the division by 7 of 10^2 leaves the same remainder as the division of the number $3 \cdot 3 = 9$, the latter remainder being equal to 2. Consequently, the division of the number $10^3 = 10^2 \cdot 10$ by 7 leaves a remainder equal to $2 \cdot 3 = 6$; the remainder resulting from the division by 7 of the number $10^6 = 10^3 \cdot 10^3$ coincides with the remainder resulting from the division of the number $6 \cdot 6 = 36$ by 7, this remainder being equal to 1. Hence, the number $10^6 = 10^3 \cdot 10^3$ can be written in the form $7k + 1$. Further, the remainders resulting from the division by 7 of the numbers $10^5 = 10^3 \cdot 10^2$ and $6 \cdot 2 = 12$ coincide and are equal to 5; in other words, the number $10^5 = 10^3 \cdot 10^2$ has the form $7l + 5$; therefore

$$10^{6n-1} = 10^{6n-6} \cdot 10^5 = (10^6)^{n-1} \cdot 10^5 =$$
$$= \underbrace{(7k+1)(7k+1)\ldots(7k+1)}_{n-1 \text{ times}}(7l+5) = 7K + 5$$

It follows that the division of the number 10^{6n-1} by 7 leaves a remainder of 5. Finally, the division of the numbers $3 \cdot 10^{6n-1}$ and $3 \cdot 5 = 15$ by 7 leaves one and the same remainder equal to 1, whence we conclude that the number $3 \cdot 10^{6n-1} - 1$ is exactly divisible by 7. The assertion stated in the problem has thus been proved.

74. The number of noughts at the end of the decimal representation of a number is equal to the maximum exponent of the power of 10 by which this number is divisible. The number 10 is equal to the product $2 \cdot 5$. The exponent of the power of 2 contained in the product of all whole numbers from 1 to 100 inclusive is greater than that of the power of 5 contained in this product. Consequently, the exponent of the highest power of 10 by which the product $1\cdot 2\cdot 3 \dots 100$ is divisible (this exponent coincides with the number of the noughts at the end of the decimal representation of the product) is equal to the exponent of the power of 5 contained in the product. Further, among the numbers from 1 to 100 there are 20 numbers multiple of five, and four among these five numbers (25, 50, 75 and 100) are also multiples of 25, that is they contain 5 to the second power. Consequently, the total number of 5's contained in the product $1\cdot 2\cdot 3 \dots 100$ is equal to 24; therefore there are exactly 24 noughts at the end of the decimal representation of this product.

75. *First solution of Problems 75* (*a*) *and* (*b*). Let us begin with Problem 75 (a). Let us denote as $t+1, t+2, \dots, t+n$ a sequence of n arbitrary consecutive whole numbers. We can determine the greatest exponent m of the power of every prime number p entering in the product $n!$ and the greatest exponent s of the power of p entering in the product $(t+1) \dots (t+n)$.

We shall denote by m_1 the number of the members in the sequence $1, 2, \dots, n$ which contain powers of p with exponents not less than 1; similarly, by m_2 we shall denote the number of the members in this sequence which contain powers of p with exponents not less than 2, etc. Then the exponent of the power of p contained in $n!$ is equal to $m = m_1 + m_2 + \dots$.

Similarly, let us denote the number of the members in the sequence $t+1, \dots, t+n$ divisible by p as s_1, the number of the members in the sequence divisible by p^2 as s_2 and so on; it is obvious that the exponent s of the power of p contained in the product $(t+1) \dots (t+n)$ is equal to $s = s_1 + s_2 + \dots$.

Further, the number of the members in the sequence $t+1, \dots, t+n$ which are divisible by p is not less than m_1. Indeed, among the numbers $t+1, \dots, t+n$ there are the numbers $t+p$, $t+2p, \dots, t+m_1p$, and in each of the intervals between $t+kp$ and $t+(k+1)p$ ($k=0, 1, 2, \dots, m_1-1$) there is at least one number divisible by p. Hence, $s_1 \geqslant m_1$; we similarly conclude that $s_2 \geqslant m_2$ etc.; therefore $s \geqslant m$. It follows that each of the prime factors of the number $n!$ is contained in the number $(t+1) \dots (t+n)$ and that the exponent of the power of each such prime factor contained in the product $(t+1) \dots (t+n)$ is not less than the exponent of the power of that prime number contained in $n!$. This means that the number $(t+1) \dots (t+n)$ is divisible by $n!$.

(b) The product of the first a factors in $n!$ coincides with $a!$; the product of the b factors following these a factors is divisible by $b!$ (see the solution of Problem 75 (a)); the product of the next c factors is divisible by $c!$ and so on. Since $a+b+c+ \dots + k \leqslant n$, it follows that $n!$ is exactly divisible by the product $a!b! \dots k!$

Second solution of Problems 75 (a) and (b). Let us begin with the solution of Problem 75 (b). As was shown, the exponent m of the power of any prime number p contained in $a!$ is equal to $m = m_1 + m_2 + \ldots$ where m_1 is the number of the members in the sequence 1, 2, ..., a which are multiple of p, the symbol m_2 indicates the number of the members in the sequence which are multiple of p^2, etc. Further, the number of the members which are multiple of p is equal to $\left[\frac{a}{p}\right]$, the number of the members multiple of p^2 is equal to $\left[\frac{a}{p^2}\right]$ etc. where $\left[\frac{a}{p}\right]$, $\left[\frac{a}{p^2}\right]$, ... are the integral parts of the fractions $\frac{a}{p}$, $\frac{a}{p^2}$, ... respectively (see page 36). Thus, $m=\left[\frac{a}{p}\right]+\left[\frac{a}{p^2}\right]+\ldots$. Now, let p be an arbitrary prime number. Then the exponent of the power of p contained in the numerator of the expression we are interested in is equal to the sum $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots$ and the exponent of the power of p contained in the denominator of this expression is equal to

$$\left[\frac{a}{p}\right]+\left[\frac{a}{p^2}\right]+\ldots+\left[\frac{b}{p}\right]+\left[\frac{b}{p^2}\right]+\ldots+\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\ldots$$

Since $n \geqslant a+b+\ldots+k$ we can use the result established in Problem 201 (1) to obtain the inequality

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots \geqslant \left(\left[\frac{a}{p}\right]+\left[\frac{b}{p}\right]+\ldots\right)+ \\ +\left(\left[\frac{a}{p^2}\right]+\left[\frac{b}{p^2}\right]+\ldots\right)+\ldots$$

This means that the exponent of the power of p in the numerator exceeds that of the power of p in the denominator. Therefore the given fraction is a whole number.

Now let us solve Problem 75 (a). To this end we multiply the product $(t+1)\ldots(t+n)$ and the expression $n!$ by the product $t(t-1)\ldots 1$ so that the former product turns into $(t+n)!$. Then the result of the division of $(t+1)\ldots(t+n)$ by $n!$ can be written as the fraction of the form

$$\frac{(t+n)\ldots(t+1)\,t\,(t-1)\ldots 1}{n!\,t\,(t-1)\ldots 1}=\frac{(n+t)!}{n!\,t!}=\frac{(t+1)\ldots(t+n)}{n!}$$

As was shown, such a fraction is in fact equal to an integral number.

(c) The expression $(n!)!$ is the product of the first $n!$ whole numbers. These $n!$ numbers can be divided into $(n-1)!$ groups each of which consists of n consecutive whole numbers. The product of the numbers forming each of these groups is divisible by $n!$, which follows from the solution of Problem 75 (a).

(d) Let the numbers in question be denoted as a, $a+d$, $a+2d, \ldots, a+(n-1)d$. We shall begin with proving that there exists a whole number k such that the division of the product kd by $n!$ leaves a remainder equal to 1. Indeed, let us consider the $n!-1$ numbers $d, 2d, 3d, \ldots, (n!-1)d$. None of them is divisible by $n!$ because d and $n!$ are relatively prime. On the other hand, there are not two products pd and qd where p and q are whole numbers which are less than $n!$ and where division by $n!$ leaves equal remainders because, if otherwise, the difference $pd-qd=(p-q)d$ would be divisible by $n!$. Thus, the division of these $n!-1$ numbers by $n!$ must leave $n!-1$ different remainders, whence it follows that there exists a number k such that the division of kd by $n!$ leaves a remainder equal to 1.

Now let us denote the product ka as A. Then we can write

$$\begin{aligned} ka &= A \\ k(a+d) &= A+kd=(A+1)+r\cdot n! \\ k(a+2d) &= A+2kd=(A+2)+2r\cdot n! \\ &\cdots\cdots\cdots\cdots\cdots \\ k[a+(n-1)d] &= A+(n-1)kd=[A+(n-1)]+(n-1)r\cdot n! \end{aligned}$$

It follows that the division by $n!$ of the product

$$k^n a(a+d)(a+2d)\ldots[a+(n-1)d]$$

leaves the same remainder as the division of the product $A(A+1)(A+2)\ldots[A+(n-1)]$. The product $A(A+1)(A+2)\ldots$ $\ldots[A+(n-1)]$ is divisible by $n!$ (see Problem 75 (a)), and the numbers k^n and $n!$ are relatively prime because, if otherwise, k would not be relatively prime to $n!$ and kd would not be relatively prime to $n!$ either.

76. The number of combinations of 1000 things, taken 500 at a time, is equal to $1000!/(500!)^2$. Since 7 is a prime number, the highest power of 7 by which $1000!$ is divisible has the exponent equal to $[1000/7]+[1000/49]+[1000/343]=142+20+2=164$ (see the second solution of Problem 75 (b)). The greatest exponent of the power 7 contained in $500!$ is equal to $[500/7]+$ $+[500/49]+[500/343]=71+10+1=82$. Consequently, the exponent of the highest power of 7 by which the denominator $(500!)^2$ is divisible is equal to $82\cdot 2=164$. Thus, both the numerator and the denominator contain 7 to the 164th power. On cancelling by this highest power of 7 we arrive at a fraction whose numerator no longer contains the factor 7, whence it follows that the whole number $(1000!)/(500!)^2$ is not divisible by 7.

77. (a) The number $(n-1)!$ is not divisible by n only in the case when n is a prime number or when $n=4$.

Indeed, if n is a composite number which can be represented as a product of some two different factors a and b then both a and b are less than $n-1$ and, consequently, a and b are contained in $(n-1)!$, whence it follows that $(n-1)!$ is divisible by $ab=n$. If n is a square of a prime number p exceeding two then $n-1=p^2-1>2p$, and therefore both p and $2p$ are contained in $(n-1)!$; consequently, $(n-1)!$ is divisible by $p \cdot 2p = 2p^2 = 2n$. Thus, the only numbers satisfying the condition of the problem are 2; 3; 4; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59; 61; 67; 71; 73; 79; 83; 89 and 97, that is these are the number 4 and all prime numbers less than 100.

(b) The number $(n-1)!$ is not divisible by n^2 only in the following cases: n is a prime number or n is a duplicated prime number or $n=8$ or $n=9$.

Indeed, if n is not a prime number and is not a duplicated prime number and is not a square of a prime number and is not equal to 8 and is not equal to 16 then n can be represented in the form $n=ab$ where a and b are different numbers not smaller than 3.

Analogously, if n is not equal to 16 then n can be written as $n=ab$ where $a \geqslant 3$ and $b \geqslant 5$. Let us assume that $n=ab$, $b>a$ and $a \geqslant 3$. Then the numbers a, b, $2a$, $2b$ and $3a$ are less than $n-1$, a, b and $2b$ being different from one another, and at least one of the numbers $2a$ and $3a$ is different from the numbers a, b and $2b$. Thus, in this case $(n-1)!$ contains as factors the numbers a, b, $2b$ and $2a$ or a, b, $2b$, and $3a$ (or perhaps all the numbers a, b, $2b$, $2a$, and $3a$). In all these cases $(n-1)!$ is divisible by $a^2b^2=n^2$.

Further, if $n=p^2$ where p is a prime number exceeding 4 then $n-1>4p$, and $(n-1)!$ contains the factors p, $2p$, $3p$ and $4p$; consequently, $(n-1)!$ is divisible by $p^4=n^2$. If $n=2p$ then the number $(n-1)!$ is not divisible by p^2 and hence it is not divisible by n^2 either; if $n=8$ or $n=9$ then $(n-1)!$ is not divisible by n^2 (7! is not divisible by 8^2 and 8! is not divisible by 9^2).

In case $n=16$ the number $(n-1)!$ is divisible by n^2 (because 15! contains the factors 2, $4=2^2$, $6=3\cdot 2$, $8=2^3$, $10=2\cdot 5$, $12=2^2\cdot 3$ and $14=2\cdot 7$, and consequently, 15! is divisible by $2^{1+2+1+3+1+2+1}=2^{11}=16^2\cdot 2^3$).

Thus, the numbers satisfying the condition of Problem 77 (b) are those satisfying the condition of Problem 77 (a) and, besides, the numbers 6, 8, 9, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86 and 94; in other words, these are all prime numbers less than 100, all duplicated prime numbers not exceeding 100 and the numbers 8 and 9.

78. Let us suppose that n is a number divisible by all numbers m less than or equal to $\sqrt{n}$. Let K be the least common multiple of all such numbers m. The factorization of the number K obviously contains all prime numbers less than $\sqrt{n}$, the exponent k of the power of each of such prime numbers p satisfying the relations $p^k \leqslant \sqrt{n}$ and $p^{k+1} > \sqrt{n}$. Let us suppose that the number of the prime numbers less than $\sqrt{n}$ is equal to l; we shall denote these prime numbers as $p_1, p_2, \ldots, p_l$. The least common multiple K of all numbers less than $\sqrt{n}$ is equal to the product $p_1^{k_1}p_2^{k_2} \ldots p_l^{k_l}$ where k_1 satisfies the inequalities $p_1^{k_1} \leqslant \sqrt{n} < p_1^{k_1+1}$, k_2 satisfies the inequalities $p_2^{k_2} \leqslant \sqrt{n} < p_2^{k_2+1}$ etc. On performing the term-by-term multiplication of the l inequalities

$$\sqrt{n} < p_1^{k_1+1}, \quad \sqrt{n} < p_2^{k_2+1}, \quad \ldots, \quad \sqrt{n} < p_l^{k_l+1}$$

we obtain

$$(\sqrt{n})^l < p_1^{k_1+1}p_2^{k_2+1} \ldots p_l^{k_l+1}$$

But we have $p_1^{k_1+1}p_2^{k_2+1} \ldots p_l^{k_l+1} = p_1^{k_1}p_2^{k_2} \ldots p_l^{k_l} \cdot p_1p_2 \ldots p_l \leqslant K^2$ because $p_1^{k_1}p_2^{k_2} \ldots p_l^{k_l} = K$, and consequently $p_1p_2 \ldots p_l \leqslant K$. Thus

$$(\sqrt{n})^l < K^2$$

According to the hypothesis, the number n must be divisible by K, and therefore we have $K \leqslant n$; consequently, $(\sqrt{n})^l < n^2$ whence it follows that $l < 4$. Since $p_1, \ldots, p_l$ are all prime numbers less than $\sqrt{n}$, there must be $p_4 = 7 > \sqrt{n}$ (the fourth prime number is equal to 7) and $n < 49$.

On investigating all numbers smaller than 49 we readily find that among them only the numbers 24, 12, 8, 6, 4 and 2 possess the required property.

79. (a) Let

$$n-2, \quad n-1, \quad n, \quad n+1, \quad n+2$$

denote five consecutive whole numbers. Then

$$(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 5n^2 + 10 = 5(n^2+2)$$

If the number $5(n^2+2)$ were a perfect square, it would be divisible by 25 and, consequently, the number n^2+2 would be divisible by 5. This is only possible when the last digit of the number n^2 is equal either to 8 or to 3, but it is known that there is no whole number whose square has 8 or 3 as its last digit.

(b) Among three consecutive whole numbers there is one number that must be divisible by 3, one number whose division by 3 leaves a remainder of 1 and one number whose division by 3 leaves a remainder equal to 2 or, which is the same, a remainder equal to —1. To the multiplication of numbers there corresponds the multiplication of the remainders resulting from the division of these numbers by a given number; indeed, we have

$$(pk + r)(qk + s) = pqk^2 + pks + qkr + rs = k(pqk + ps + qr) + rs$$

Therefore, if the division of a number by 3 leaves a remainder of 1 then the remainder resulting from the division of any power of this number by 3 is also equal to 1. In case the division of a number by 3 leaves a remainder equal to —1, the division of any odd power of that number by 3 leaves a remainder of —1 and the division by 3 of its any even power leaves a remainder of 1.

Thus, among three even powers of consecutive whole numbers there is one divisible by 3 while the remainders resulting from the division by 3 of the other two powers are equal to 1. Consequently, the division of a sum of even powers of three consecutive whole numbers leaves a remainder equal to 2 or, which is the same, a remainder equal to —1. However, as was already shown, such a remainder cannot result from the division by 3 of an even power of any whole number.

Remark. It should be noted that in the above proof we do not use the fact that the powers to which three consecutive numbers are raised have *equal* even exponents. Therefore there holds the following more general assertion: *a sum of even powers* (which may have different exponents) *of three consecutive whole numbers cannot be equal to an even power of any whole number.*

(c) As was shown in the solution of Problem 79 (b), a sum of three even powers of consecutive whole numbers by 3 leaves a remainder 2. It follows that the division by 3 of a sum of even powers of nine consecutive whole numbers leaves a remainder equal to $2 + 2 + 2 = 6$, which simply means that the sum is divisible by 3. Now let us prove that a sum of powers of nine consecutive whole numbers with equal even exponents cannot be divisible by $3^2 = 9$; the assertion of the problem is obviously an immediate consequence of the last proposition.

Among nine consecutive whole numbers there is one number which must be divisible by 9, one number whose division by 9 leaves a remainder equal to 1, one number whose division by 9 leaves a remainder equal to 2 and so on. It follows that if the even exponent of the power to which the nine numbers are raised is equal to $2k$ then the division of the sum under consideration and of the sum

$$0 + 1^{2k} + 2^{2k} + 3^{2k} + 4^{2k} + 5^{2k} + 6^{2k} + 7^{2k} + 8^{2k}$$

leaves one and the same remainder. It is also clear that the division of the latter sum and of the expression

$$2(1^k + 4^k + 7^k)$$

by 9 leaves one and the same remainder because the numbers 3^2 and 6^2 are divisible by 9, the division of the numbers 1^2 and $8^2 = 64$ by 9 leaves the same remainder equal to 1, the division of the numbers $2^2 = 4$ and $7^2 = 49$ by 9 leaves a remainder equal to 4 and the division of the numbers $4^2 = 16$ and $5^2 = 25$ by 9 leaves a remainder equal to 7.

Now we note that the division of the numbers $1^3 = 1$, $4^3 = 64$ and $7^3 = 343$ by 9 leaves the same remainder equal to 1, whence it follows that if $k = 3l$ then the division of the sums $1^k + 4^k + 7^k = 1^l + 64^l + 343^l$ and $1^l + 1^l + 1^l = 3$ by 9 leaves the same remainder (equal to 3). Hence, the former sum is not divisible by 9. Similarly, it follows that if $k = 3l + 1$ then the division of $1^k + 4^k + 7^k = 1^l \cdot 1 + 64^l \cdot 4 + 343^l \cdot 7$ by 9 leaves the same remainder as the division of the sum $1 \cdot 1 + 1 \cdot 4 + 1 \cdot 7 = 12$, that is the expression $1^k + 4^k + 7^k$ is not divisible by 9, and if $k = 3l + 2$ then the division by 9 of the sum $1^k + 4^k + 7^k = 1^l \cdot 1 + 64^l \cdot 4^2 + 343^l \cdot 7^2$ and of the sum $1 \cdot 1 + 1 \cdot 16 + 1 \cdot 49 = 66$ leaves the same remainder, which means that in this case $1^k + 4^k + 7^k$ is not divisible by 9 either.

80. (a) The sum of the digits of each of the numbers A and B is equal to

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

whence it follows that the division of both numbers by 9 leaves a remainder 1 (the division of every number by 9 leaves the same remainder as the division by 9 of the sum of the digits of the number). If we had $A/B = n$ or, which is the same, $A = nB$ where n is a whole number different from 1 then the relation $B = 9N + 1$ would imply $A = nB = 9M + n$, which means that the division of n by 9 would leave a remainder of 1. But the smallest number n possessing this property is equal to 10 whereas we bave $A/B < 10$ because both A and B are 7-digit numbers. This contradiction shows that A cannot be divisible by B.

(b) Let N, $2N$ and $3N$ denote the sought-for numbers. The division of a whole number by 9 leaves a remainder equal to the one resulting from the division by 9 of the sum of its digits. Therefore the division of the sum $N + 2N + 3N$ by 9 leaves the same remainder as the division by 9 of the sum $1 + 2 + 3 + \ldots + 9 = 45$, whence it follows that $6N$ and, consequently, $3N$ are divisible by 9.

Since $3N$ is a three-digit number, the initial digit of the number N cannot exceed 3; therefore the last digit of the number N

cannot be equal to 1 (because, if otherwise, the last digit of $2N$ would be equal to 2, the last digit of $3N$ would be equal to 3 and hence the initial digit of N could not be smaller than 3). The last digit of the number N cannot be equal to 5 either because, if otherwise, the number $2N$ would end with nought. Now let us suppose that the last digit of the number N is equal to 2; in this case the last digits of the numbers $2N$ and $3N$ are equal to 4 and 6 respectively. Therefore the first two digits of $3N$ can only assume the values 1, 3, 5, 7, 8, and 9; since the sum of all digits of the number $3N$ is multiple of 9 the first two digits of $3N$ can only be equal to 3 and 9 or to 5 and 7. On testing all the possible cases we find one triple of numbers satisfying the condition of the problem for the case when the last digit of N is equal to 2: 192; 384; 576. The cases when the last digit of N is equal to 3, 4, 6, 7, 8 or 9 are investigated in like manner; it turns out that there are three more solutions of the problem: the triple 273; 546; 819, the triple 327; 654; 981 and the triple 219; 438; 657.

81. A perfect square can only have 0, 1, 4, 9, 6 or 5 as its last digit. Further, the square of every even number is obviously divisible by 4 while the division of the square of an odd number by 4 must leave a remainder equal to 1 because $(2k)^2 = 4k^2$ and $(2k+1)^2 = 4(k^2+k)+1$. Therefore there is no whole number whose square has 11, 99, 66 and 55 as its last two digits because the division of a number ending with the digits 11, 99, 66 or 55 by 4 leaves remainders equal to 3, 3, 2 and 3 respectively. Now let us consider the remainders resulting from the division of the squares of whole numbers by 16. Every whole number can be written in one of the following five forms:

$$8k, \quad 8k \pm 1, \quad 8k \pm 2, \quad 8k \pm 3 \quad \text{and} \quad 8k+4$$

Accordingly, the squares of these expressions are

$$16 \cdot (4k^2), \quad 16(4k^2 \pm k)+1, \quad 16(4k^2 \pm 2k)+4,$$
$$16(4k^2 \pm 3k)+9 \quad \text{and} \quad 16(4k^2+4k+1)$$

Thus, we see that either the square of a whole number is divisible by 16 or its division by 16 leaves a remainder equal to 1, 4 or 9. As to the numbers whose last four digits are 4444, their division by 16 leaves a remainder of 12, and consequently these numbers cannot be perfect squares.

Thus, if a perfect square ends with four identical digits, these digits can only be four noughts (for instance, $100^2 = 10\,000$).

82. Let x, y and z denote the lengths of the sides and of the diagonal of the rectangle respectively. Then, by Pythagoras' theorem, we have

$$x^2 + y^2 = z^2$$

It is required to prove that the product xy is divisible by 12. We shall first show that this product is divisible by 3 and then that it is divisible by 4.

Since

$$(3k+1)^2 = 3(3k^2+2k)+1 \quad \text{and} \quad (3k+2)^2 = 3(3k^2+4k+1)+1$$

we see that the division by 3 of the square of any number which is not multiple of 3 leaves a remainder equal to 1. Consequently, if neither x nor y were divisible by 3, the division of the sum x^2+y^2 by 3 would leave a remainder equal to 2, and therefore the sum x^2+y^2 could not be equal to a square of a whole number. Hence, if x^2+y^2 is equal to the square of an integer z then at least one of the numbers x and y is divisible by 3 and thus xy is divisible by 3.

Further, it is clear that the numbers x and y cannot be simultaneously odd; for, if $x=2m+1$ and $u=2n+1$, then the expression

$$x^2+y^2 = 4m^2+4m+1+4n^2+4n+1 = 4(m^2+m+n^2+n)+2$$

cannot be equal to the square of a whole number because the square of an odd number is itself odd and the square of an even number must be divisible by 4. In case both x and y are even numbers their product is of course divisible by 4. Let us suppose that x is even and y is odd: $x=2m$ and $y=2n+1$. In this case the number z is odd (because $z^2=x^2+y^2$ is odd), that is $z = 2p+1$. Then we have

$$(2m)^2 = (2p+1)^2-(2n+1)^2 = 4p^2+4p+1-4n^2-4n-1$$

that is

$$m^2 = p(p+1)-n(n+1)$$

It follows that m^2 is even (because the products $p(p+1)$ and $n(n+1)$ of two consecutive whole numbers must be even). Consequently, the number m is even and the number $x=2m$ is divisible by 4. Thus, in this case as well the product xy is divisible by 4.

83. By the formula for the roots of a quadratic equation, we have

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Consequently, for the roots of the equation to be rational numbers it is necessary and sufficient that the expression b^2-4ac should be a perfect square. Let us put $b=2n+1$, $a=2p+1$ and

$c = 2q + 1$; then we have

$$b^2 - 4ac = (2n+1)^2 - 4(2p+1)(2q+1) = 4n^2 + 4n - \\ - 16pq - 8p - 8q - 3 = 8\left(\frac{n(n+1)}{2} - 2pq - p - q - 1\right) + 5$$

Since the number $b^2 - 4ac$ is odd (because $n(n+1)$ is a product of two consecutive whole numbers and therefore it is an even number, whence it follows that $n(n+1)/2$ is a whole number), we see that if $b^2 - 4ac$ is a square of a whole number then this whole number must be odd. Every odd number can be represented in the form $4k \pm 1$ and its square can be written in the form

$$(4k \pm 1)^2 = 16k^2 \pm 8k + 1 = 8(2k^2 \pm k) + 1$$

Consequently, the division of the expression $(4k \pm 1)^2$ by 8 always leaves a remainder equal to 1. Therefore, since the division of the number $b^2 - 4ac$ by 8 leaves a remainder of 5, the expression $b^2 - 4ac$ cannot be a perfect square.

84. We have

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} = \frac{3n^2 + 6n + 2}{n(n+1)(n+2)}$$

The numerator of the fraction on the right-hand side is not divisible by 3 while its denominator is exactly divisible by 3 because it is equal to the product of three consecutive whole numbers. Consequently, the denominator always contains prime factors different from 2 and 5, and therefore when this fraction is written in decimal notation we obtain an *infinite repeating* decimal.

Among the two whole numbers n and $n+1$ there must be one which is even. If $n+1$ is even then n is odd and, consequently, $3n^2$ is odd, whence it follows that the entire numerator is an odd number. If n is even then $n+2$ is also even (that is, it is divisible by 2) and, consequently, the denominator must be divisible by 2^2 whereas the numerator is divisible by 2 and is not divisible by 2^2 because for $n = 2k$ we have

$$3n^2 + 6n + 2 = 12k^2 + 12k + 2 = 2(6k^2 + 6k + 1)$$

Hence, after the given fraction has been reduced to its lowest terms its denominator is not relatively prime to 10 and therefore in the decimal representation of the fraction we obtain a *mixed* periodic decimal.

85. (a), (b) Let us reduce all the fractions in the sum

$$M = \frac{1}{2} + \ldots + \frac{1}{n}$$

and in the sum

$$N = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+m}$$

to common denominators. From all fractions in the sums let us choose the ones whose denominators contain the highest powers of two (there can be only one such fraction in each of the sums). When these fractions are reduced to the common denominators their numerators and denominators are multiplied by the corresponding odd factors. When the other fractions in the sums are reduced to the common denominators the factors by which their numerators and denominators are multiplied must contain the number 2. On adding together all the fractions in the sums M and N we obtain fractions whose denominators are of course even numbers, and each of their numerators is a sum of several even numbers and one odd number, which follows from what was said about the reduction of the fractions to the common denominators. Consequently, the numerators of the resultant fractions are odd numbers, and therefore these fractions cannot be equal to whole numbers.

(c) Among the fractions in the sum K let us choose the one whose denominator contains the highest power of the number 3 (let the exponent of this power of 3 be k). Since the denominators of all fractions are odd numbers, the sum K does not contain the fraction $1/2\cdot 3^k$. Therefore when we reduce all these fractions in the sum to the common denominator the factor by which the denominator and the numerator of the fraction we have chosen are multiplied is not divisible by 3 whereas for the other fractions such factors are divisible by 3. Consequently, on adding together all the fractions in the sum we arrive at a fraction whose denominator is divisible by 3 and whose numerator is not divisible by 3, whence it follows that K cannot be equal to a whole number.

86. (a) The fractions $(a^3+2a)/(a^4+3a^2+1)$ and $(a^4+3a^2+1)/(a^3+2a)=a+\frac{a^2+1}{a^3+2a}$ (and also the fraction $(a^2+1)/(a^3+2a)$) are simultaneously reducible or irreducible. Further, the fractions $(a^2+1)/(a^3+2a)$ and $(a^3+2a)/(a^2+1)=a+a/(a^2+1)$ (and also the fraction $a/(a^2+1)$) are simultaneously reducible or irreducible as well. Finally the fractions $a/(a^2+1)$ and $(a^2+1)/a=a+1/a$ (and the fraction $1/a$) are also simultaneously reducible or irreducible. Now, to complete the solution of the problem it is sufficient to note that for any integral a the fraction $1/a$ cannot be reduced by a factor.

(b) If both the number $a=5n+6$ and the number $b=8n+7$ are divisible by an integer $d\neq 1$ then the difference $p=b-a=3n+1$, the difference $p_1=a-p=2n+5$ and also the differences $p_2=p-p_1=n-4$; $p_3=p_1-p_2=n+9$ and $p_4=p_3-p_2=13$ are also divisible by d. Thus, the (prime) number 13 must necessarily be divisible by d, whence it follows that d can only assume one value 13. The fact that the case when the given fraction is reducible by 13 is possible can readily be demonstrated by an example; for instance, if $n = 4$ then the fraction $(5n + 6)/(8n + 7) = 26/39$ can be reduced by 13.

87. It is clear that it is sufficient to prove the assertion of the problem under the assumption that we start reading the digits of the new number beginning with the *second* digit of the number obtained in the initial reading. Indeed, on performing such "shifts" by one digit an appropriate number of times we can pass, in succession, from any given initial digit to any other digit beginning with which we read the corresponding 1953-digit number. If after every such "shift" we pass from a number divisible by 27 to another number which is also divisible by 27 then this divisibility is retained for any number of the shifts. Let the initial digit of the first 1953-digit number we have read be $a_1 = a$; let the 1952-digit number formed of the other digits contained in that 1953-number be denoted as B. Then the first number is equal to $a \cdot 10^{1952} + B$, and the new 1953-digit number which we read beginning with the second digit of the former 1953-digit number (this is initial digit of the number B) is equal to $B \cdot 10 + a$.

According to the condition of the problem, the number $a \cdot 10^{1952} + B$ is divisible by 27. The number $10^{1952} - 1$ is written with the aid of 1952 nines; on dividing this number by 9 we obtain a number consisting of 1952 ones, which, being divided by 3, yields a remainder which is equal to the remainder resulting from the division by 3 of the sum of the digits of that number. Since this sum of the digits is equal to 1952, the remainder we speak of is equal to 2. Consequently, the division of the number $10^{1952} - 1$ by 27 leaves a remainder equal to $2 \cdot 9 = 18$ and the division of 10^{1952} by 27 leaves a remainder equal to 19. Therefore, on denoting as b the remainder resulting from the division of the number B by 27, we conclude that the division of the numbers $a \cdot 10^{1952} + B$ and $19a + b$ by 27 leaves coinciding remainders. Hence, the condition of the problem implies that the number $M = 19a + b$ is divisible by 27.

Now let us pass to the new number $B \cdot 10 + a$. The remainders resulting from the division of the numbers $B \cdot 10 + a$ and $N = 10b + a$ by 27 are equal, and hence the problem reduces to the proof of the fact that if $M = 19a + b$ is divisible by 27 then $N = 10b + a$ is also divisible by 27. But this is quite obvious because $10M - N = 189a$ is divisible by 27 (since we have $189 = 27 \cdot 7$) and $N = 10 \cdot M - 189a$.

88. It is obvious that the last digit in the decimal notation of every number of the form 5^n (where n is a natural number) is equal to 5, and therefore the decimal representation of the number $a = 5^{1000}$ also ends with the digit 5 (but not with 0). Now let us suppose that the decimal representation of the number a contains noughts as well. Suppose that the first (counting from right to left) of these noughts occupies the ith place. It is clear that the decimal representation of the number $5^{1000}\cdot 10^{i-1}$ ends with the digits $5\underbrace{000\ldots 0}_{i-1 \text{ noughts}}$; consequently, the last $i-1$ digits in the decimal representation of the sum $a_1 = 5^{1000} + 5^{1000}\cdot 10^{i-1}$ coincide with the $i-1$ last digits of the number a (that is, all these $i-1$ digits are different from nought), and the ith digit (counting from right to left) of the number a_1 is also equal to 5 (that is, it is also different from 0). Thus, we have replaced the number a by the number a_1 whose decimal representation contains a greater number of digits and whose ith digit (counting from right to left) is different from 0. Next we perform the same operation on the number a_1: if its decimal representation contains noughts and the first of these noughts (counting from right to left) occupies the jth place where, of course, $j > i$, then we replace a_1 by the number $a_2 = a_1 + 5^{1000}\cdot 10^{j-1}$ which is also divisible by 5^{1000}, the last j digits of a_2 being different from 0 (its jth digit is equal to 5; here again the digits are counted from right to left).

If the continuation of this process results in a number a_k (where k can be equal to 0, 1, 2, ... and by a_0 is meant the number $a = 5^{1000}$ itself) whose decimal representation does not contain noughts, the assertion of the problem turns out to be true. However, since the number of the digits in the decimal representations of $a_0 = a$, a_1, a_2, ... permanently increases, the process may last indefinitely. In this case we can stop the process when we arrive at a number a_l whose last 1000 digits are different from 0 because the number a_l can be written as $a_l = 10^{1000}\cdot A + B$ where the decimal representation of the number B and the decimal representation of the number a_l have the same last 1000 digits; consequently *all the digits of B are different from* 0 and the decimal representation of the number A consists of all the digits of the number a_l preceding the 1000th digit (counting from right to left) of a_l, whence it follows that

$$B = a_l - 10^{1000}\cdot A$$

We see that the number B is divisible by 5^{1000} (because both a_l and $10^{1000}\cdot A = 2^{1000}\cdot 5^{1000}\cdot A$ are divisible by 5^{1000}).

89. The numbers forming the sequence we are interested in are all of the form $1 + 10^4 + 10^8 + \ldots + 10^{4k}$ ($k=1, 2, \ldots$). Let us also consider the numbers of the form $1 + 10^2 + 10^4 + 10^6 + \ldots$ $\ldots + 10^{2k}$. We can verify directly that

$$10^{4k+4} - 1 = (10^4 - 1)\cdot(1 + 10^4 + 10^8 + \ldots + 10^{4k})$$

and

$$10^{2k+2} - 1 = (10^2 - 1)\cdot(1 + 10^2 + 10^4 + \ldots + 10^{2k})$$

Besides, we obviously have

$$10^{4k+4}-1=\left(10^{2k+2}-1\right)\left(10^{2k+2}+1\right)$$

From these three equalities we derive the relation

$$10^{4k+4}-1=(10^4-1)(1+10^4+10^8+\ldots+10^{k})=$$
$$=(10^2-1)(1+10^2+10^4+\ldots+10^{2k})\left(10^{2k+2}+1\right)$$

whence, since $(10^4-1)/(10^2-1)=10^2+1=101$, we obtain

$$(1+10^4+10^8+\ldots+10^{4k})\cdot 101=$$
$$=(1+10^2+10^4+\ldots+10^{2k})\left(10^{2k+2}+1\right)$$

Since 101 is a prime number, we see that $1+10^2+10^4+\ldots+10^{2k}$ or $10^{2k+2}+1$ is divisible by 101. In case $k>1$ the quotient resulting from the division by 101 of $1+10^2+10^4+\ldots+10^{2k}$ or of $10^{2k+2}+1$ exceeds 1. On cancelling the last relation by 101 we conclude that for $k>1$ the number $1+10^4+10^8+\ldots+10^{4k}$ can be written as a product of at least two factors, which is what we intended to prove. For $k=1$ we have the number $10^4+1=10\,001$ which is also composite ($10\,001=73\cdot 137$).

Remark. In just the same way we can prove that all the numbers forming the sequence

$$\underbrace{100\ldots0}_{(2k+1)\text{ times}}\quad \underbrace{100\ldots0}_{(2k+1)\text{ times}}1;\quad \underbrace{100\ldots0}_{(2k+1)\text{ times}}\quad \underbrace{100\ldots0}_{(2k+1)\text{ times}}\quad \underbrace{100\ldots0}_{(2k+1)\text{ times}}1;\ \ldots$$

are composite.

90. Using the formula $a^2-b^2=(a+b)(a-b)$ we can write

$$2^{2^n}-1=\left(2^{2^{n-1}}+1\right)\left(2^{2^{n-1}}-1\right)=$$
$$=\left(2^{2^{n-1}}+1\right)\left(2^{2^{n-2}}+1\right)\left(2^{2^{n-2}}-1\right)=\ldots$$
$$\ldots=\left(2^{2^{n-1}}+1\right)\left(2^{2^{n-2}}+1\right)\left(2^{2^{n-3}}+1\right)\ldots(2^2+1)(2+1)$$

(the last factor $2-1=1$ has been dropped). Thus, the number $2^{2^n}-1=\left(2^{2^n}+1\right)-2$ is divisible by all the preceding numbers of the sequence in question. It follows that if the numbers $2^{2^n}+1$ and $2^{2^k}+1$ (where $k<n$) have a common factor then the number 2 must also be divisible by that factor. However, the number 2 cannot be a common factor of two numbers belonging to the sequence since all these numbers are odd; consequently, every two numbers belonging to the sequence are relatively prime.

91. The number 2^n cannot of course be divisible by 3. If the division of 2^n by 3 leaves a remainder of 1 then 2^n-1 is divisible by 3; if the division of 2^n by 3 leaves a remainder of 2 then

2^n+1 is divisible by 3. Consequently, in all the cases one of the two numbers 2^n-1 and 2^n+1 is divisible by 3, and hence if both these numbers exceed 3, they cannot be simultaneously prime numbers.

92. (a) If the remainder resulting from the division of a prime number $p>3$ by 3 were equal to 2 then $8p-1$ would be divisible by 3. Therefore if $8p-1$ is a prime number then the division of p by 3 must leave a remainder equal to 1; in this case $8p+1$ is divisible by 3; if p is equal to 3 the number $8p+1=25$ is also composite.

(b) If p is not divisible by 3, the division of p^2 by 3 leaves a remainder of 1 (see the solution of Problem 79 (b)), and consequently $8p^2+1$ is divisible by 3. Thus, we must have $p=3$ and $8p^2+1=73$. We see that in this case $8p^2-1=71$ is also a prime number.

93. The division of a prime number different from 2 and 3 by 6 leaves a remainder equal to 1 or 5 because if the remainder resulting from the division of that number by 6 were equal to 2 or 4, this prime number would be even (which is impossible because the prime number is supposed to be different from 2) and if the remainder were equal to 3 the prime number would be divisible by 3 (which is also impossible). Thus, any prime number exceeding 3 can be written in the form $6n+1$ or $6n+5$. The squares of these expressions are equal to $36n^2+12n+1$ and $36n^2+60n+25$ respectively, and in both cases the remainders resulting from the division of these squares by 12 are equal to 1.

94. The three prime numbers in question are of the form $6n+1$ or $6n+5$ (see the foregoing problem). Therefore at least two of them have the same form. Consequently, the difference of these two prime numbers which is equal to d or to $2d$ where d is the common difference of the progression is divisible by 6. Hence, d is divisible by 3. Besides, since d is a difference of two odd numbers, it must be divisible by 2. Therefore d is divisible by 6. (Also see the solution of Problem 95 (a).)

95. (a) Since all prime numbers different from 2 are odd, the common difference of the progression is an even number. Further, if the common difference of the progression were not divisible by 3 then the division by 3 of the three terms a_1, a_1+d and a_1+2d of the progression would leave different remainders because among them there are not two numbers whose difference is not divisible by 3. Consequently, at least one of them would be divisible by 3, which is impossible because, according to the condition of the problem, all the terms of the progression are prime numbers (if $a_1=3$ then a_1+3d is also divisible by 3). Similarly, if d were not divisible by 5, the division of all the numbers a_1, a_1+d, a_1+2d, a_1+3d and a_1+4d by 5 would leave different remainders and, consequently, at least one of them would be divisible But this inequality cannot hold for $k \geqslant 2$, whence it follows that all the terms of the arithmetic progression are prime numbers implies that the common difference of the progression must be divisible by 7. Thus, the common difference d of the sought-for progression must be multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$, that is $d = 210k$.

By the condition of the problem, we have

$$a_{10} = a_1 + 9d = a_1 + 1890k < 3000$$

But this inequality cannot hold for $k \geqslant 2$, whence it follows that $k = 1$. Therefore $a_1 < 3000 - 9d = 1110$.

Further, we have $210 = 11 \cdot 19 + 1$, and consequently the $(m + 1)$th term of the progression can be written in the form

$$a_{m+1} = a_1 + (11 \cdot 19 + 1) \cdot m = 11 \cdot 19m + (a_1 + m)$$

It follows that if the division of a_1 by 11 leaves a remainder of 2 then a_{10} is divisible by 11; if the remainder resulting from the division of a_1 by 11 is equal to 3 then a_9 is divisible by 11, etc. Thus, we can prove that the division of a_1 by 11 cannot leave a remainder equal to 2, 3, 4, ... or 10. If the number a_1 is different from 11 then it cannot be divisible by 11 (because a_1 is a prime number). Hence, either a_1 is equal to 11 or the division of a_1 by 11 leaves a remainder equal to 1. Further, using the equality $210 = 13 \cdot 16 + 2$ which implies

$$a_{m+1} = a_1 + (13 \cdot 16 + 2)\, m = 13 \cdot 16m + (a_1 + 2m)$$

we can show that the division of a_1 by 13 leaves a remainder which can only be equal to 2, 4, 6, 8, 10 and 12. Now, taking into account that the number a_1 is odd (because all the terms of the progression are odd), we conclude that it is either equal to 11 or can be written in one of the following forms:

$$2 \cdot 11 \cdot 13l + 23 = 286l + 23, \quad 286l + 45, \quad 286l + 67,$$
$$286l + 155, \quad 286l + 177 \quad \text{and} \quad 286l + 199$$

Since $a_1 < 1110$, it only remains to test the following values of a_1:

11; 23; 309; 595; 881; 45; 331; 615; 903; 67;
353; 637; 925; 155; 441; 727; 1013; 177; 463;
749; 1035; 199; 485; 771; 1057

Among these values only the numbers

11; 23; 881; 331; 67; 353; 727; 1013; 463 and 199

are prime.

On testing the 10 progression corresponding to these prime numbers we find that only one of them satisfies the conditions of the problem, namely this is the arithmetic progression

199; 409; 619; 829; 1039; 1249; 1459; 1669; 1879; 2089

(b) This problem is solved by analogy with Problem 95 (a). First of all, by analogy with the solution of Problem 95 (a), we conclude that if a_1 is different from 11 then the common difference of the progression must be proportional to $2\cdot 3\cdot 5\cdot 7\cdot 11 = 2310$, that is $d = 2310k$. It follows that

$$a_{11} = a_1 + 23\,100k > 20\,000$$

Now it remains to consider the value $a_1 = 11$. In this case we can only assert that $d = 210k$. Using the equality $210 = 13\cdot 16 + 2$ we can write the following expression for the general term of the progression:

$$a_{n+1} = 11 + (13\cdot 16 + 2)\,kn = 13\,(16kn + 1) + 2\,(kn - 1)$$

Further, for each of the numbers $k = 1, 2, 3, 4, 5, 7, 8, 9, 10$ it is possible to indicate a number $n < 10$ such that $kn - 1$ is divisible by 13, and consequently a_{n+1} is divisible by 13 and is not a prime number. The values of n corresponding to the enumerated values of k are equal to 1, 7, 9, 10, 8, 2, 5, 3, 4 respectively. In the case when $k = 6$ we have $d = 210\cdot 6 = 1260$, and the term

$$a_4 = 11 + 3\cdot 1260 = 3791$$

is divisible by 17. Thus, if $a_1 = 11$ then $k > 10$, and consequently $d \geqslant 2100$, whence we again obtain $a_{10} > 20\,000$.

Remark. In the solutions of Problems 95 (a) and (b) we used the term "prime number" in the ordinary sense and assumed that the terms of the progression were positive. If this assumption is dropped and by a "prime number" is meant any integer n which has no divisors different from ± 1 and $\pm n$ then there exist arithmetic progressions satisfying the conditons of Problem 95 (b) (here is an example of such a progression: —11; 199; 409; 619; 829; 1039; 1249; 1459; 1669; 1879; 2089).

96. (a) If two numbers are not equal to each other and differ by not more than 4, they cannot have common divisors exceeding 4. Thus, two of the given five consecutive whole numbers can have 2 or 3 or as their common divisor or can be relatively prime to each other. Further, among five consecutive whole numbers there are at least two odd numbers and among two consecutive odd numbers at least one is not divisible by 3. Consequently, among the given numbers there is at least one odd number not divisible by 3 and this number must be relatively prime to the other four numbers.

(b) The solution of this problem resembles the solution of Problem 96 (a) but is much more complicated. If two numbers are not equal to each other and differ by not more than 15, they cannot have common divisors greater than 15. Since two numbers having no common prime divisors are relatively prime, we see that in order to prove the theorem it is sufficient to show that among 16 consecutive whole numbers there must be one having no common divisors equal to 2, 3, 5, 7, and 13 with the other 15 numbers; this number is relatively prime to all the others.

We shall begin with deleting eight even numbers from the given 16 numbers because they do not satisfy the necessary requirements; the remaining numbers form a sequence of eight consecutive odd numbers. Among eight consecutive odd numbers the 1st, the 4th and the 7th or the 2nd, the 5th and the 8th or the 3rd and the 6th must obviously be divisible by 3; further, the 1st and the 6th or the 2nd and the 7th or the 3rd and the 8th or only one of these numbers are divisible by 5. Similarly, the 1st and the 8th or only one of the eight consecutive odd numbers are divisible by 7 whereas there is not more than one number divisible by 11 or 13. If among the eight consecutive odd numbers there are not more than five numbers divisible by 3 or by 5 or by 7 then there is a number among them which is not divisible by 3, by 5, by 7, by 11 and by 13; this number must be relatively prime to the other numbers. Now we shall consider all the cases when there are not less than 6 numbers divisible by 3 or by 5 or by 7.

Suppose that there are three numbers among the eight consecutive odd numbers which are divisible by 3; then two of the remaining numbers can be divisible by 5 only if one of the extreme numbers (that is the smallest or the greatest of the numbers) is divisible by 3 while the other is divisible by 5. On deleting these five numbers we arrive at a sequence consisting of the 2nd, the 5th and the 6th numbers or of the 7th, the 4th and the 3rd numbers. Let us begin with the first case. The 2nd, the 5th and the 6th odd numbers occupy the 4th, the 10th and the 12th or the 3rd, the 9th and the 11th places in the original sequence of all 16 consecutive whole numbers. None of these numbers can have a common divisor equal to 13 with the other 15 numbers because each of these 15 numbers differs from that number by less than 13. Consequently, if these three numbers are divisible neither by 3 nor by 5 then one of them (namely, the one which is divisible neither by 7 nor by 11) is relatively prime to all the other numbers. The proof is carried out in just the same way in the case when after the numbers divisible by 3 or by 5 have been deleted there remain the 3rd, the 4th and the 7th numbers.

If among the eight numbers under consideration there are three numbers divisible by 3 then there are not two numbers among the remaining five numbers which are divisible by 7. In case there are only two numbers, the 3rd and the 6th ones, divisible by 3 then there can be two numbers among the remaining numbers, namely the 1st and the 8th, divisible by 7, and two numbers, the 2nd and the 7th, divisible by 5. On deleting these six numbers we obtain a pair consisting of the 4th and the 5th of the eight odd numbers which are not divisible by 3, by 5 and by 7. These two numbers are relatively prime to each of the 15 remaining numbers of the given sequence because each of the remaining numbers differs from these two numbers by less than 11 and cannot therefore have a common divisor equal to 11 or 13 with those two numbers.

Thus, we have completed the proof of the assertion that among any 16 consecutive whole numbers there is always one number relatively prime to the others.

Remark. In a similar but simpler way it can be proved that among 8 or among 10 or among any number *less than* 16 of consecutive whole numbers there is always one number relatively prime to the others. For 17 consecutive numbers this assertion no longer holds: for instance, among the 17 consecutive numbers from 1184 to 1200 there is no number relatively prime to all the others. For any other number k exceeding 16 it is probably possible to inidicate k consecutive whole numbers among which there is no number relatively prime to the others; as far as we know, no proof of this general assertion has yet been elaborated.

97. The sought-for number is equal to the product $A_1 \cdot B_1$ where A_1 consists of 666 digits 9 and B_1 of 666 digits 2. The number A_1 is less by 1 than the number 10^{666} whose decimal representation consists of one digit 1 and 666 noughts; the product of the number B_1 by A_1 is equal to the product of B_1 by 10^{666} (the latter product consists of 666 twos and 666 noughts) minus the number B_1. It easily be seen that this difference has the form $\underbrace{22\ldots2}_{665 \text{ times}}1\underbrace{77\ldots7}_{665 \text{ times}}8$.

98. The number 777 777 is exactly divisible by 1001 and the quotient resulting from the division is equal to 777. Therefore the division of the number $\underbrace{777\ldots7}_{996 \text{ times}}00000$ by 1001 gives the quotient

$$\underbrace{777000777000\ldots77700000}_{\text{the group 777000 is repeated 166 times}}$$

Besides, since the quotient resulting from the division of the number 77 777 by 1001 is equal to 77 and the remainder is equal to 700, the division of A by 1001 gives the quotient

$$\underbrace{777000777000\ldots77700077}_{\text{the group 777000 is repeated 166 times}}$$

and the remainder 700.

99. Since the number 222 222 is not a perfect square the decimal representation of the sought-for number has the form $222\,222\,a_7a_8\ldots a_n$ where $a_7, a_8, \ldots, a_n$ are some unknown digits.

Let us first suppose that the number n of the digits of the sought-for number is even: $n = 2k$. On extracting the square root of the sought-for number according to ordinary rules we obtain

$$\sqrt{22\,22\,22\,a_7a_8\ldots a_{2k-1}a_{2k}} = 471\,405$$

$$\begin{array}{r|l}
 & 16 \\
87 & 6\,22 \\
7 & 6\,09 \\
\hline
941 & 13\,22 \\
1 & 9\,41 \\
\hline
9424 & 3\,81\,a_7a_8 \\
4 & 3\,76\,9\,6 \\
\hline
942805 & x_1x_2x_3a_9a_{10}a_{11}a_{12} \\
5 & 4\;7\;1\;4\;0\;2\;5 \\
\hline
\end{array}$$

(the fifth digit of the result is equal to 0 since the digit x_1 can obviously be equal only to 4 or 5 and, consequently, it must be less than 9; by the same reason, in case the sixth digit is the last one, it must be equal to 5).

The remainder is equal to zero if $a_9 = 4$, $a_{10} = 0$, $a_{11} = 2$, $a_{12} = 5$, $x_1 = 4$, $x_2 = 7$ and $x_3 = 1$, whence we readily find that $a_8 = 6 + 1 = 7$ and $a_7 = (7 + 9) - 10 = 6$. Thus, the smallest number consisting of an even number of digits and satisfying the condition of the problem is equal to $222\,222\,674\,025 = 471\,405^2$.

The case when the number n is odd ($n = 2k + 1$) is considered in an analogous manner:

$$\sqrt{2\,22\,22\,2a_7a_8a_9\ldots a_{2k}a_{2k+1}} = 149071\ldots$$

$$\begin{array}{r|l}
 & 1 \\
24 & 1\,22 \\
4 & 96 \\
\hline
289 & 26\,22 \\
9 & 26\,01 \\
\hline
29807 & 21\,2a_7a_8a_9 \\
7 & 20\,89\,4\,9 \\
\hline
298141 & x_1x_2x_3x_4a_{10}a_{11} \\
1 & 2\;9\;8\;1\;4\;1 \\
\hline
298142 & x_5x_6x_7x_8x_9x_{10}a_{12}a_{13}
\end{array}$$

Since the number consisting of the two digits x_1 and x_2 is not less than $33 = 119 - 86$ and not greater than $43 = 129 - 86$, the sixth digit of the root is equal to 1, and the extraction of the root does not stop and must be continued. Consequently, the smallest number consisting of an odd number of digits and satisfying the condition of the problem has not less than thirteen digits, that is it exceeds the number 222 222 674 025.

Thus, the sought-for number is equal to 222 222 674 025.

100. The equality $m^2 = n^2 + 1954$ implies that the numbers m^2 and n^2 *are simultaneously even or odd.* Consequently, so are the numbers m and n. Therefore the number $1954 = m^2 - n^2 = (m + n)(m - n)$ must necessarily be divisible by 4 (because both numbers $m + n$ and $m - n$ are even) whereas 1954 is not divisible by 4. Hence, the numbers m and n satisfying the condition of the problem do not exist.

101. The sought-for six-digit number which begins with the digits 523 and is exactly divisible by $7 \cdot 8 \cdot 9 = 504$ can be represented in the form $523\,000 + X$ where X is a three-digit number. The division of 523 000 by 504 shows that $523\,000 = 504 \cdot 1037 + 352$, that is the division of 523 000 by 504 leaves a remainder equal to 352. Since the sum of the number 523 000 and the three-digit number X must be divisible by 504, the number X can be equal either to

$$504 - 352 = 152$$

or to

$$2 \cdot 504 - 352 = 656$$

(because $3 \cdot 504 - 352$ is a four-digit number). Thus, there are two numbers satisfying the condition of the problem: 523 152 and 523 656.

102. Let N be the sought-for number. By the condition of the problem, we have

$$N = 131k + 112 = 132l + 98$$

where k and l are positive integers. Besides, since N is a four-digit number, there hold the inequalities

$$l = \frac{N - 98}{132} < \frac{10\,000 - 98}{132} \leqslant 75$$

Further, we have

$$131k + 112 = 132l + 98 \quad \text{whence} \quad 131(k - l) = l - 14$$

It follows that if $k - l$ is different from zero then the absolute value of $l - 14$ exceeds 130, which is impossible when $l \leqslant 75$.

Thus, there must be $k-l=0$, that is $k=l$, whence we readily obtain

$$l-14=0, \quad k=l=14$$

and

$$N=131\cdot 14+112=(132\cdot 14+98)=1946$$

103. (a) The $2n$-digit number indicated in the condition of the problem can be transformed in the following way:

$$4\cdot 10^{2n-1}+9\cdot 10^{2n-2}+4\cdot 10^{2n-3}+9(10^{2n-4}+10^{2n-5}+\ldots$$
$$\ldots+10^{n})+5\cdot 10^{n-1}+5\cdot 10^{n-2}=4\cdot 10^{2n-1}+9\cdot 10^{2n-2}+$$
$$+4\cdot 10^{2n-3}+9\cdot 10^{n}\frac{10^{n-3}-1}{9}+5\cdot 10^{n-1}+5\cdot 10^{n-2}=$$
$$=4\cdot 10^{2n-1}+9\cdot 10^{2n-2}+5\cdot 10^{2n-3}-10^{n}+5\cdot 10^{n-1}+5\cdot 10^{n-2}=$$
$$=\frac{1}{2}(8\cdot 10^{2n-1}+18\cdot 10^{2n-2}+10\cdot 10^{2n-3}-2\cdot 10^{n}+10^{n}+10^{n-1})=$$
$$=\frac{1}{2}(9\cdot 10^{2n-1}+9\cdot 10^{2n-2}-9\cdot 10^{n-1})=\frac{[(10^{n}-1)+10^{n-1}]\cdot 9\cdot 10^{n-1}}{2}$$

This expression is equal to the sum of the terms of an arithmetic progression whose common difference is 1, the first term is 10^{n-1} and the last term is $10^{n}-1$ (the number of the terms of the progression is equal to $10^{n}-10^{n-1}=9\cdot 10^{n-1}$), that is this expression is equal to the sum of all n-digit numbers.

(b) The number of those of the numbers in question whose initial digit is a (a can be equal to 1, 2, 3, 4 or 5) is equal to $6\cdot 6\cdot 3=108$ (since the second and the third digits can be equal to any of the six digits 0, 1, 2, 3, 4 and 5 while the fourth digit can be equal to any of the *three* digits 0, 2 and 4 because we consider only even numbers). It follows that the total sum of all thousands the integral number of which is contained in all the numbers in question is equal to $(1+2+3+4+5)\cdot 108\cdot 1000=1\,620\,000$.

Analogously, the collection of those numbers whose second digit assumes a fixed value is equal to $5\cdot 6\cdot 3=90$ because the initial digit can be equal to one of the *five* digits 1, 2, 3, 4 and 5. It follows that the total sum of hundreds the integral number of which is contained in all the numbers in question (minus the sum of the integral number of thousands) is equal to $(1+2+3+4+5)\times 90\cdot 100=135\,000$.

In just the same manner we find that the total sum of tens the integral number of which is contained in the numbers in question is equal to $(1+2+3+4+5)\cdot 90\cdot 10=13\,500$ and, finally, the total sum of ones is equal to $(2+4)\cdot 5\cdot 6\cdot 6\cdot 1=1080$.

Thus, the sought-for sum is

$$1\,620\,000 + 135\,000 + 13\,500 + 1080 = 1\,769\,580$$

104. Let us first consider the integers from 0 to 99 999 999; we shall complete those of the integers which consists of less than eight digits with a number of noughts on the left to make them contain eight digits each. Then we shall have 100 000 000 eight-digit numbers for whose representation we obviously need 800 000 000 digits. It is evident that each of the 10 digits will be used the same number of times because they all are quite equivalent since 0 may occupy the initial place like any other digit. Consequently, every digit will be used 80 000 000 times.

Now let us find the number of the extra noughts, that is the number of the noughts written on the left of the numbers consisting of less than eight digits. There are only nine 1-digit numbers (here we do not take into account the number 0), $99 - 9 = 90$ two-digit numbers, $999 - 99 = 900$ three-digit numbers etc. Since to every 1-digit number we added seven noughts on the left, to every two-digit number we added six noughts etc., the total number of the extra noughts (we do not take into account the digits of the first number which is written as 00 000 000 in this notation) is equal to

$$7 \cdot 9 + 6 \cdot 90 + 5 \cdot 900 + 4 \cdot 9000 + 3 \cdot 90\,000 + 2 \cdot 900\,000 + \\ + 1 \cdot 9\,000\,000 = 11\,111\,103$$

Now let us write 1 on the left of the first number 00 000 000; then we obtain all whole numbers from 1 to 100 000 000. We see that for the decimal representation of these numbers we need 80 000 000 twos, threes etc. up to nines, 80 000 001 ones (one extra digit 1 was written on the left of 00 000 000) and $80\,000\,000 - 11\,111\,103 = 68\,888\,897$ noughts.

105. There are exactly nine one-digit numbers, $99 - 9 = 90$ two-digit numbers, $999 - 99 = 900$ three-digit numbers and, generally, $9 \cdot 10^{n-1}$ n-digit numbers.

The one-digit numbers occupy nine places in the sequence under consideration, the digits of the two-digit numbers occupy $90 \cdot 2 = 180$ places, the digits of the three-digit numbers occupy $900 \cdot 3 = 2700$ places, the digits of the four-digit numbers occupy $9000 \cdot 4 = 36\,000$ places and the digits of the five-digit numbers occupy $90\,000 \cdot 5 = 450\,000$ places, whence it is seen that the digit we are interested in belongs to a five-digit number.

The digits belonging to the numbers consisting of not more than four digits occupy the places with indices from 1 to

$9+180+2700+36\,000=38\,889$ inclusive. To determine the number of the five-digit numbers which lie in the interval from the 38 889th place to the 206 788th place in the sequence in question we must divide the difference $206\,788-38\,889=167\,899$ by 5, which results in a quotient of 33 579 and in a remainder of 4, that is

$$206\,788-38\,889=5\cdot 33\,579+4$$

Thus, the sought-for digit belongs to the 33 580th five-digit number, that is to the number 43 579, because the first five-digit number is 10 000. In this number the digit we are interested in is the fourth one (counting from left to right), and consequently it is equal to 7.

106. Let us suppose that 0.1234 ... is a periodic decimal whose period consists of n digits, the number of the digits preceding the period being equal to k. Let us consider the number $N=10^m$ where m is an integer not smaller than $n+k$; in the decimal notation this number is written as 1 with m noughts following it. When forming the decimal under consideration we consecutively write all the *whole* numbers; consequently the number N is also placed somewhere after the decimal point. Now, since we have supposed that the decimal 0.1234 ... is periodic and since there are $m\geqslant n+k$ noughts standing side by side somewhere in the decimal 0.1234 ..., we conclude that the period of the decimal consists of noughts only, which is obviously impossible. Thus the decimal 0.1234 ... *cannot be periodic.*

107. As is well known, the division of an arbitrary positive integer N by 9 leaves a remainder equal to the sum of the digits of the integer (this follows from the fact that the digit a_k occupying the $(k+1)$th decimal place in the decimal representation of the number N symbolizes the term $a_k\cdot 10^k$ in the expansion of N in powers of ten; as to the number $a_k\cdot 10^k=a_k\cdot(99\ldots 9+1)=$ $=99\ldots 9a_k+a_k$, its division by 9 leaves the same remainder as a_k). It follows that if the remainder r resulting from the division of the number N by 9 is different from zero then after a sufficient number of the replacements of the numbers by the sums of their digits have been performed we eventually arrive at the one-digit number r and if N is divisible by 9 we arrive at the number 9. Thus, among the 1 000 000 000 one-digit numbers resulting from the operations we have performed the ones correspond to the numbers 1; 10; 19; 28; ... ; 1 000 000 000 whose division by 9 leaves a remainder of 1 and the twos correspond to the numbers 2; 11; 20; 29; ... ; 999 999 992 whose division by 9 leaves a remainder of 2. It is evident that the number of the numbers in the former group exceeds by unity that in the latter group, and therefore the *number of ones exceeds by unity the number of twos.*

108. **(a)** The answer to the question is negative. For, if the decimal representation of a number $N = n^2$ ends with nought, it must necessarily have an even number of noughts at the end, and the number N_1 obtained from N by deleting these noughts must also be a perfect square. Therefore it is sufficient to prove that a number whose decimal representation consists only of the digits 6 and 0 and which ends with the digit 6 cannot be a perfect square. Indeed, such a number has either 06 or 66 as its last two digits, that is it is *divisible by* 2 *and is not divisible by* 4; hence, it *cannot* be a perfect square.

(b) The answer to the question is negative. For, if the decimal representation of a number $N = n^2$ ends with the digit 5, then n must also end with the same digit, that is $n = 10n_1 + 5$ and $n^2 = (10n_1 + 5)^2 = 100n_1^2 + 100n_1 + 25 = 100n_1(n_1 + 1) + 25$. Consequently, the last two digits of the number $N = 100N_1 + 25$ are 25, and we have $N_1 = n_1(n_1 + 1)$. It is clear that the last digit of the expression $n_1(n_1 + 1)$ coincides with the last digit of the product of the last digits of the numbers n_1 and $n_1 + 1$; this digit can only be equal to 0, 2 or 6 because the products $0 \cdot 1$; $4 \cdot 5$; $5 \cdot 6$ and $9 \cdot 0$ end with the digit 0, the products $1 \cdot 2$; $3 \cdot 4$; $6 \cdot 7$ and $8 \cdot 9$ end with the digit 2 and the products $2 \cdot 3$ and $7 \cdot 8$ with the digit 6. Thus, the last digit of the number N_1 and the third digit (counting from right to left) of the number N must necessarily be equal to 6 because neither the combination of digits 025 nor the combination 225 can stand at the end of the number N (the decimal representation of N does not contain the digit 0, and the digit 2 enters into it only once). Since we have $N = 1000N_2 + 625$, the number N is divisible by $5^3 = 125$ because both $1000N_2$ and 625 are divisible by 125; therefore, since N is a perfect square, the number N must also be divisible by $5^4 = 625$. However, since N is divisible by 5^4, the number $1000N_2 = N - 625$ is also divisible by 5^4 and consequently N_2 must be divisible by 5. Now, since the number N_2 is divisible by 5, its last digit can only be 0 or 5; this means that the decimal representation of the number N has 0625 or 5625 as its last four digits, which is impossible because the decimal representation of N contains only one digit 5 and does not contain 0 at all.

109. To prove the assertion of the problem it is sufficient to show that the 444 445-digit number A which is obtained when we consecutively write 88 889 five-digit numbers $a_1, a_2, a_3, \ldots, a_{88\,889}$ is divisible by a whole number different from 2. It can easily be shown that *A is divisible by* 11 111. Indeed, the number $A = \overline{a_1 a_2 a_3 \ldots a_{88\,887} a_{88\,888} a_{88\,889}}$ (here the bar indicates that A is the number written with the aid of the digits forming the numbers

$a_1, a_2, a_3, \ldots, a_{88\,887}, a_{88\,888}, a_{88\,889})$ can obviously be written as

$$A = a_{88\,889} + a_{88\,888}\cdot 10^5 + a_{88\,887}\cdot 10^{10} + \ldots + a_1\cdot 10^{44\,440} =$$
$$= (a_1 + a_2 + \ldots + a_{88\,888} + a_{88\,889}) + a_{88\,888}(10^5 - 1) +$$
$$+ a_{88\,887}(10^{10} - 1) + \ldots + a_1(10^{44\,440} - 1) \quad (*)$$

Now, since $a_1, a_2, \ldots, a_{88\,889}$ are *all* whole numbers from 11 111 to 99 999 inclusive, the sum of these numbers is equal to

$$11\,111 + 11\,112 + 11\,113 + \ldots + 99\,999 =$$
$$= \frac{11\,111 + 99\,999}{2}\cdot 88\,889 = 11\,111\cdot 5\cdot 88\,889$$

and hence it is divisible by 11 111. All the other addends on the right-hand side of formula (*) are also divisible by 11 111 because, for any integral k, the difference $(10^5)^k - 1^k$ between the powers kth of 10^5 and 1 is divisible by the difference of the bases $10^5 - 1 = 99\,999 = 9\cdot 11\,111$ and hence it is divisible by 11 111. Therefore A is divisible by 11 111.

110. Let the sought-for number be $X = \overline{a_0a_1a_2\ldots a_9}$ where $a_0, a_1, \ldots, a_9$ are the digits of the number and the bar designates the number itself. According to the condition of the problem, a_0 is equal to the number of noughts among the digits of the number X, a_1 is equal to the number of ones, a_2 is equal to the number of twos etc. Therefore the sum of all digits of X is equal to

$$a_0 + a_1 + a_2 + \ldots + a_9 = a_0\cdot 0 + a_1\cdot 1 + a_2\cdot 2 + \ldots + a_9\cdot 9$$

whence we obtain

$$a_0 = a_2 + 2a_3 + 3a_4 + 4a_5 + 5a_6 + 6a_7 + 7a_8 + 8a_9 \quad (*)$$

By the conditions of the problem, $a_0 \neq 0$ because, if otherwise, X would not be a 10-digit number (by the way, the condition $a_0 \neq 0$ is readily implied by (*)). If $a_0 = 1$ then we must have $a_0 = a_2 = 1$ and $a_1 = 8$ (because the total number of the digits must be equal to 10), and all the other digits must be equal to 0, which is impossible. If $a_0 = 2$ then we must have $a_0 = a_2 = 2$ and $a_1 = 6$, and all the other digits must be equal to 0 or we must have $a_0 = 2$, $a_3 = 1$, and $a_1 = 7$, and all the other digits must be equal to 0, which is impossible either. Now, let $a_0 = i > 2$. Equality (*) can be rewritten in the form

$$a_0 = i = a_2 + 2a_3 + \ldots + (i-1)a_i + \ldots + 8a_9 \quad (**)$$

(if $i = 3$ then the terms $2a_3$ and $(i-1)a_i$ coincide; if $i = 9$ then $(i-1)a_i$ and $8a_9$ coincide). Here a_i is the number of the digits of X equal to i; therefore $a_i \neq 0$ because $a_0 = i$; on the other hand, equality (**) cannot hold for $a_i > 1$, and hence $a_i = 1$. Therefore (**) can be rewritten thus:

$$1 = a_2 + 2a_3 + \dots + (i-2)a_{i-1} + ia_{i+1} + \dots + 8a_9 \quad (***)$$

From (***) it readily follows that $a_2 = 1$ and that all the digits of the number X different from a_0, a_1, a_2 and a_i are equal to 0. Since $a_2 = 1$, there is a digit equal to 2 among the digits of the number X, and it is evident that only the digit a_1 can be equal to 2. Thus, in the decimal representation of X only the digits $a_0 = i$, $a_1 = 2$, $a_2 = 1$ and $a_i = 1$ are different from 0, that is there are i noughts, 2 ones, 1 digit equal to two and one digit equal to i among the digits of the number X, whence in follows that, since X is a 10-digit number, $i = 10 - 2 - 1 - 1 = 6$.

Thus, $X = 6\,210\,001\,000$ (it can easily be verified that this number does in fact satisfy all the conditions of the problem).

111. Since $A = 999\,999\,999 = 1\,000\,000\,000 - 1$, the product AX can be expressed as $AX = 10^9X - X = \overline{x_1x_2\dots x_k\,000\,000\,000} - \overline{x_1x_2\dots x_k}$ where $X = \overline{x_1x_2\dots x_k}$ is an arbitrary natural number (written with the aid of the digits $x_1, x_2, \dots, x_k$; the bars designate the numbers consisting of the corresponding digits). It is required that the number AX should have only ones in its decimal representation, that is

$$\overline{x_1x_2\dots x_k\,000\,000\,000} - \overline{x_1x_2\dots x_k} = \overline{11\dots1111}$$

whence

$$\overline{x_1x_2\dots x_k\,000\,000\,000} - \overline{11\dots1111} = \overline{x_1x_2\dots x_k}$$

Since all the digits of the minuend in the last relation are known we can write the numbers in a column to perform the subtraction and to determine, in succession, all the digits of the number X beginning with the last one:

$$\begin{array}{r} \overbrace{(7)(8)\dots(8)(8)}^{8\text{ digits}}(9) \\ 000\;\;000\;\;000 \\ -\dots11\dots111\;\;111\;\;111\;\;111 \\ \hline \underbrace{\dots67\dots777}_{9\text{ digits}}\;\;888\;\;888\;\;889 \end{array}$$

Here we have written in the parentheses the digits of the number X which are determined consecutively. This process should be continued until we arrive at a group of ones which mutually cancel when we form the difference; indeed, the decimal representations of the number X in the minuend and in the subtrahend can be made to coincide only when the number X obtained in the difference begins with a group of noughts (which, of course, are not taken into account).

Finally, we arrive at the following value of the number $X = X_0$:

$$X_0 = \underbrace{11\ldots1}_{9\text{ digits}}\underbrace{22\ldots2}_{9\text{ digits}}\underbrace{33\ldots3}_{9\text{ digits}}\ldots\underbrace{77\ldots7}_{9\text{ digits}}\underbrace{88\ldots8}_{9\text{ digits}}9$$

The decimal representation of this number consists of $\underbrace{9+9+\ldots+9}_{7\text{ summands}}+8+1 = 72$ digits.

The number X_0 we have obtained is obviously the *smallest* of all the numbers possessing the required property and there are also other numbers possessing the same property. The matter is that we have stopped the process of subtraction when a group of nine digits 0 has appeared at the beginning of the number X obtained as the difference (we have simply discarded these noughts because they have appeared at the beginning of the decimal representation of the number X). However, we can also continue the subtraction process after these nine noughts have appeared; to this end we simply write these noughts after the digits 1 in the minuend. It is clear that on continuing the subtraction we shall again obtain the digits forming the number X_0 which will stand on the left of the nine noughts; the decimal representation of X will then take the form

$$X_1 = \overline{\overline{X}_0\underbrace{00\ldots0}_{9\text{ digits}}\overline{X}_0}$$

where $\overline{X}_0$ symbolizes the number X_0 in the decimal notation. Generally, our argument shows that all the numbers X satisfying the condition of the problem are of the form

$$X = X_n = \overline{\overline{X}_0\underbrace{00\ldots0}_{9\text{ digits}}\overline{X}_0\underbrace{00\ldots0}_{9\text{ digits}}\overline{X}_0\underbrace{00\ldots0}_{9\text{ digits}}\overline{X}_0\ldots\overline{X}_0\underbrace{00\ldots0}_{9\text{ digits}}\overline{X}_0}$$

where the number n of the groups consisting of nine consecutive noughts can be quite arbitrary, that is $n = 0, 1, 2, \ldots$. The number X_n consists of $72(n+1)+9n = 81n+72$ digits.

112. Let $A = \overline{a_n a_{n-1}\ldots a_2 a_1}$ be an n-digit number. We can of course suppose that $a_1 \neq 0$ because the noughts at the end of the decimal representation of the number A can be simply discarded since this does not change the sum of the digits of the number AN for any N. Now let us consider a number $N = 10^m - 1 = \underbrace{999\ldots9}_{m\text{ nines}}$ where m / n. It is evident that the sum of the digits of the number

$$AN = 10^m A - A = \overline{a_n a_{n-1} \dots a_1 \underbrace{000 \dots 0}_{m \text{ noughts}}} - \overline{a_n a_{n-1} \dots a_1} =$$

$$= \overline{a_n a_{n-1} \dots a_2 (a_1 - 1) \underbrace{999 \dots 9}_{m-n \text{ nines}} (9-a_n)(9-a_{n-1}) \dots (9-a_2)(10-a_1)}$$

coincides with the sum of the digits of the number N which is equal to $9m$.

113. First of all we note that the assertion stated in the problem can also be considered true for $m = 0$ (in this case the stipulation that $n \geqslant 2$ becomes unnecessary and we can take the value $n = 1$ as well) on condition that we put $0^0 = 1$ (since $a^0 = 1$ for all a). For instance, let $n = 1$; on denoting all the even digits (including the digit 0) as $\alpha_1, \alpha_2, \dots$ and the odd digits as $\beta_1, \beta_2, \dots$ we can write

$$\alpha_1^0 + \alpha_2^0 + \dots = 0^0 + 2^0 + 4^0 + 6^0 + 8^0 = 1 + 1 + 1 + 1 + 1 = 5$$

and

$$\beta_1^0 + \beta_2^0 + \dots = 1^0 + 3^0 + 5^0 + 7^0 + 9^0 = 1 + 1 + 1 + 1 + 1 = 5$$

Now let us pass to the next value of n: let $n = 2$; by $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ we shall again denote all even and all odd digits respectively and by $A_1, A_2, \dots$ and $B_1, B_2, \dots$ we shall denote all the numbers which can be formed of not more than two digits for which the sum of the digits is even or odd respectively. It is evident that every number A_i has the form $\overline{\alpha_k \alpha_l} = 10\alpha_k + \alpha_l$ or the form $\overline{\beta_k \beta_l} = 10\beta_k + \beta_l$ (the bars designate the numbers consisting of the corresponding digits) and every number B_j has the form $\overline{\alpha_k \beta_l} = 10\alpha_k + \beta_l$ or $\overline{\beta_k \alpha_l} = 10\beta_k + \alpha_l$. Denoting the sums of all numbers A_i and of all numbers B_j as $\sum A_i$ and $\sum B_j$ respectively and the sums of all numbers α_k and of all numbers β_l as $\sum \alpha_k$ and $\sum \beta_l$ respectively we can write

$$\sum A_i = (10\alpha_k + \alpha_l) + \sum (10\beta_k + \beta_l) =$$
$$= 10\left(\sum \alpha_k + \sum \beta_k\right) + \left(\sum \alpha_l + \sum \beta_l\right)$$

and, analogously,

$$\sum B_j = \sum (10\alpha_k + \beta_l) + \sum (10\beta_k + \alpha_l) =$$
$$= 10\left(\sum \alpha_k + \sum \beta_k\right) + \left(\sum \beta_l + \sum \alpha_l\right)$$

The last two relations show that $\sum A_i = \sum B_j$.

In just the same manner we can prove the required assertion for the general case of an arbitrary n using the *method of mathe-*

matical induction. To this end we suppose that the assertion has already been proved for a certain natural number n (and for all $m < n$) and then prove that it remains true for the next natural number $n+1$ (and for an arbitrary exponent $m < n+1$). Let us agree to denote all nonnegative numbers consisting of not more than n digits for which the sums of the digits are even or odd as $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ respectively; further, by analogy with the above, we shall denote all the numbers consisting of not more than $n+1$ digits for which the sums of the digits are even or odd as $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ respectively. For the even and the odd digits we shall again use the notation $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ respectively. By the induction hypothesis, we have

$$\sum_k a_k^p = \sum_k b_k^p = s_{n,\,p} \quad \text{for all} \quad p < n \tag{*}$$

Further, let us denote as

$$S_{n,\,p} = \sum_k a_k^p + \sum_k b_k^p \tag{**}$$

the sum of the pth powers of *all* numbers whose decimal representations involve not more than n digits.

Now we note that every number A_i has the form $\overline{a_k\alpha_l} = 10a_k + \alpha_l$ or $\overline{b_k\beta_l} = 10b_k + \beta_l$; similarly, every number B_j has the form $\overline{a_k\beta_l} = 10a_k + \beta_l$ or $\overline{b_k\alpha_l} = 10b_k + \alpha_l$. Using the notation analogous to the above we find that

$$\sum_i A_i^m = \sum (10a_k + \alpha_l)^m + \sum (10b_k + \beta_l)^m \quad \text{where} \quad m < n+1$$

For the expression $(10a_k + \alpha_l)^m$ we can write, using Newton's binomial formula, the relation

$$(10a_k + \alpha_l)^m = 10^m \cdot a_k^m + C(m,\,1) \cdot 10^{m-1} a_k^{m-1}\alpha_l + \\ + C(m,\,2) \cdot 10^{m-2} a_k^{m-2}\alpha_l^2 + \ldots + C(m,\,m-1) \cdot 10 a_k \alpha_l^{m-1} + \alpha_l^m$$

The expression $(10b_k + \beta_l)^m$ can be written in the same manner. Therefore, using (*) we obtain

$$\sum (10a_k + \alpha_l)^m = 10^m \sum a_k^m + C(m,\,1) \cdot 10^{m-1} \sum a_k^{m-1} \sum \alpha_l + \\ + C(m,\,2) \cdot 10^{m-2} \sum a_k^{m-2} \sum \alpha_l^2 + \ldots \\ \ldots + C(m,\,m-1) \cdot 10 \sum a_k \sum \alpha_l^{m-1} + \sum \alpha_l^m = \\ = 10^m \sum a_k^m + C(m,\,1) \cdot 10^{m-1} s_{n,\,m-1} \sum \alpha_l + \\ + C(m,\,2) \cdot 10^{m-2} s_{n,\,m-2} \sum \alpha_l^2 + \ldots \\ \ldots + C(m,\,m-1)\, 10 s_{n,\,1} \sum \alpha_l^{m-1} + \sum \alpha_l^m$$

and, similarly,

$$\sum (10b_k + \beta_l)^m = 10^m \sum b_k^m + C(m,\ 1) \cdot 10^{m-1} \sum b_k^{m-1} \sum \beta_l + $$
$$+ C(m,\ 2) \cdot 10^{m-2} \sum b_k^{m-2} \sum \beta_l^2 + \ldots$$
$$\ldots + C(m,\ m-1) \cdot 10 \sum b_k \sum \beta_l^{m-1} + \sum \beta_l^m =$$
$$= 10^m \sum b_k^m + C(m,\ 1) \cdot 10^{m-1} s_{n,\ m-1} \sum \beta_l +$$
$$+ C(m,\ 2) \cdot 10^{m-2} s_{n,\ m-2} \sum \beta_l^2 + \ldots$$
$$\ldots + C(m,\ m-1) \cdot 10 s_{n,\ 1} \sum \beta_l^{m-1} + \sum \beta_l^m$$

Thus,

$$\sum A_i^m = \sum (10a_k + \alpha_l)^m + \sum (10b_k + \beta_l)^m =$$
$$= 10^m S_{n,\ m} + C(m,\ 1) \cdot 10^{m-1} s_{m,\ m-1} S_{1,\ 1} +$$
$$+ C(m,\ 2) \cdot 10^{m-2} s_{n,\ m-2} S_{1,\ 2} + \ldots$$
$$\ldots + C(m,\ m-1) \cdot 10 s_{n,\ 1} S_{1,\ m-1} + S_{1,\ m}$$

Now, using (*), (**) and Newton's binomial formula and performing analogous transformations we arrive at exactly *the same* expression for the sum

$$\sum B_j^m = \sum (10a_k + \beta_l)^m + \sum (10b_k + \alpha_l)^m$$

which completes the proof of the assertion.

For the exponents $m \geqslant n$ the identity indicated in the condition of the problem does not hold. For instance, if we take $n = m = 1$ then

$$\sum \alpha_i^1 = \sum \alpha_i = 0 + 2 + 4 + 6 + 8 = 20$$

whereas

$$\sum \beta_j^1 = \sum \beta_j = 1 + 3 + 5 + 7 + 9 = 25$$

114. Let us "truncate" the given triangular table in the following way: we discard the first two horizontal rows and in every following row we leave the first four numbers. Further, let us symbolize every even number by the letter e and every odd number by the letter o. Then we arrive at a table of the form

$$\begin{array}{cccccccccc}
 & & & & & o & e & o & e \\
 & & & & o & o & e & o & \\
 & & & o & e & e & e & & \\
 & & o & o & o & e & & & \\
 & o & e & o & e & & & & \\
 \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & &
\end{array}$$

We see that the fifth row of the new table coincides with the first one. Further, each of the first four numbers in every horizontal row of the given number triangle is even or odd depending solely on whether the first four numbers of the preceding row are even or odd. Consequently, in the new table the rows must repeat periodically with a period of four rows. Since each of the first four rows of the new table contains an even number it follows that all the other rows contain even numbers as well.

115. Let us prove that *the sum of the numbers in every horizontal row of the given table is divisible by* $1958/2 = 979$, *the sum of the numbers in every row beginning with the second one even being divisible by* 1958. This auxiliary proposition implies the assertion stated in the problem because the lowermost "row" contains only one number and the "sum of the numbers" in this row coincides with that number.

It is clear that the sum S_1 of the numbers in the uppermost row (which is equal to the sum of the first 1958 natural numbers forming an arithmetic progression whose common difference is equal to 1) is equal to $1/2\ (1958 \cdot 1959)$, that is S_1 is divisible by 1958/2. Further, the sum S_2 of the numbers in the second row can obviously be written as $S_2 = (0+1)+(1+2)+(2+3)+\ldots+$ $+(1956+1957)+(1957+1958) = 2S_1 - (0+1958)$ since each of the terms of the sum S_1 is involved in the expression of S_2 exactly twice with the exception of the "extreme" terms 0 and 1958 each of which is involved in the expression of S_2 only once. Now, since both $2S_1$ and $0+1958 = 1958$ are divisible by 1958 so is the sum S_2.

Let us pass to the third row of the table. It is evident that the sum S_3 of all numbers contained in this row can be written as

$$S_3 = (1+3)+(3+5)+\ \ldots\ +(3913+3915) = 2S_2 - (1+3915)$$

Indeed, every number belonging to the second row is involved exactly twice in the expression of S_3 except the "extreme" numbers 1 and 3915 each of which is involved in the sum S_3 exactly once. These numbers 1 and 3915 belonging to the second row can be written as $1 = 0+1$ and $3915 = 1957+1958$; therefore the sum

$$1+3915 = (0+1)+(1957+1958) =$$
$$= (0+1958)+(1+1957) = 2\cdot 1958$$

is divisible by 1958, whence it follows that the sum S_3 is also divisible by 1958.

This argument can be continued. The sum S_4 of all numbers belonging to the fourth row is equal to

$$S_4 = 2S_3 - (4+7828)$$

where $4 = 1 + 3$ and $7828 = 3913 + 3915$ are the "extreme" numbers of the third row, and the rule according to which the given table is formed readily implies that the sum

$$4 + 7828 = (1 + 3) + (3913 + 3915) = (0 + 1) + (1 + 2) + (1956 + 1957) + (1957 + 1958) = (0 + 1958) + (1 + 1957) + (1 + 1957) + (2 + 1956) = 4 \cdot 1958$$

is divisible by 1958 whence it follows that the sum S_4 is divisible by 1958. In exactly the same way we prove that the sum of the numbers contained in every other row is also divisible by 1958 (the formal proof can be carried out using the method of mathematical induction), which completes the proof of the assertion stated in the problem.

116. If a pole reads $\overline{xyz}|\overline{x_1y_1z_1}$ (where x, y, ... are digits and the bars designate the numbers consisting of the corresponding digits) then $\overline{x_1y_1z_1} = 999 - \overline{xyz}$ and, consequently, $z_1 = 9 - z$, $y_1 = 9 - y$ and $x_1 = 9 - x$. (If $x = 9$ or $x = y = 9$ then the digits $x_1 = 0$ or $x_1 = y_1 = 0$ are not written on the pole.) It follows immediately that if $x = y = z$ (this means that we also have $x_1 = y_1 = z_1 = 9 - x$) then the conditions of the problem hold; in this case we have 10 poles satisfying the condition of the problem (they correspond to the distances of 0 = 000; 111; 222; ... and 999 km from the poles to station A). In the case when the number $\overline{xyz}$ is written with the aid of two different digits these digits must be such that their sum is equal to 9; then the number $\overline{x_1y_1z_1} = \overline{9-x, 9-y, 9-z}$ is written with the aid of the same digits. There obviously are five pairs of such digits: (0; 9), (1; 8), (2; 7), (3; 6) and (4; 5). Further, for two given digits a and b there are six numbers whose decimal representations consist of these digits: three of them contain two digits a and one digit b (the latter digit can occupy any of the three places) and, similarly, the other three contain one digit a and two digits b. Thus we see that there are $5 \cdot 6 = 30$ more poles satisfying the condition of the problem imposed on the distances from the poles to station A. Hence, the total number of the poles satisfying the required conditions is equal to $10 + 30 = 40$.

117. *First solution.* The time interval from the beginning of the first show to the end of the seventh show is less than 13 hours because the first show begins not earlier than at 12 hours and the seventh show ends earlier than at one hour in the morning; the interval from the beginning of the second performance to the end of the sixth show is greater than 9 hours since the second performance begins before 14 hours and the sixth performance ends not earlier than at 23 hours. Consequently, since 13 hours/7 < 1 hour 52 minutes and 9 hours/5 = 1 hour 48 minutes, the interval from the beginning of a show to the beginning of the next show is shorter than 1 hour 52 minutes and is longer than 1 hour 48 minutes. Usually the duration of a show is expressed by a number of minutes multiple to 5. Let us assume in this first solution that this is the case. Then it follows that the duration of every performance is 1 hour 50 minutes.

Further, since the first performance ends before 14 hours it begins either at 12 hours 00 minutes or at 12 hours 05 minutes; accordingly, the second performance begins at 13 hours 50 minutes or at 13 hours 55 minutes and so on.

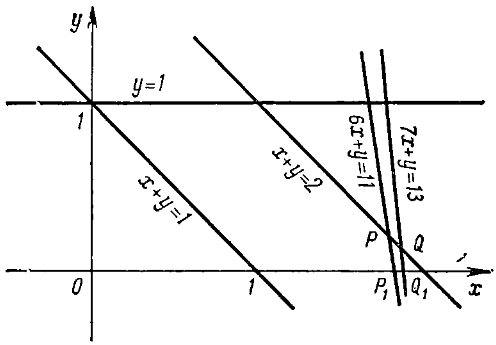


Fig. 11

If we do not introduce the requirement that the number of minutes every show lasts should be multiple of five then there are more than two solutions of the problem. This case is considered below.

Second solution. Let $12 + y$ be the time of the beginning of the first show and let x be the duration of every show (here it is meant that x and y are expressed in hours). Then

the 1st show begins at $12 + y$ hours (between 12 hours and 13 hours)
the 2nd show begins at $12 + y + x$ hours (between 13 hours and 14 hours)
the 7th show begins at $12 + y + 6x$ hours (between 23 hours and 24 hours)
the 8th show begins at $12 + y + 7x$ hours (between 24 hours and 1 hour in the morning)

whence it follows that

$$12 \leqslant 12 + y < 13, \quad \text{that is} \quad 0 \leqslant y < 1$$
$$13 \leqslant 12 + x + y < 14, \quad \text{that is} \quad 1 \leqslant x + y < 2$$
$$23 \leqslant 12 + 6x + y < 24, \quad \text{that is} \quad 11 \leqslant 6x + y < 12$$
$$24 \leqslant 12 + 7x + y < 25, \quad \text{that is} \quad 12 \leqslant 7x + y < 13$$

On discarding those of the inequalities which follow from the other inequalities we arrive at the system of inequalities

$$0 \leqslant y < 1; \quad 1 \leqslant x + y < 2; \quad 11 \leqslant 6x + y; \quad 7x + y < 13$$

which can be easily solved graphically as is shown in Fig. 11. Any point belonging to the quadrilateral PQQ_1P_1 shaded in the figure represents a solution of the problem.

118. Since the trains approach the crossing at n hours and at n hours 38 minutes, the hour hand of a timepiece (see Fig. 12)

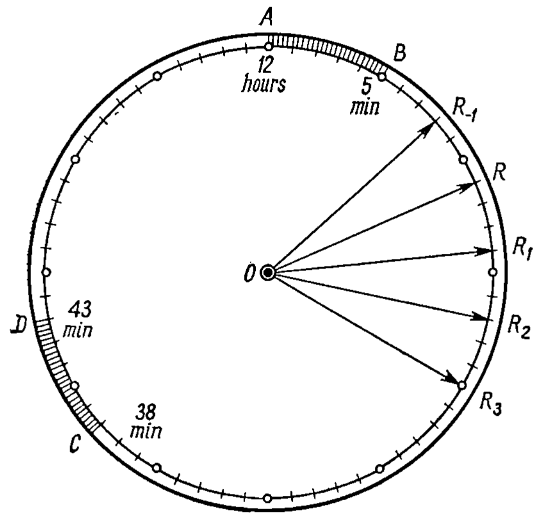


Fig. 12

occupies the position from 0 minutes to 5 minutes and from 38 minutes to 43 minutes when the lifting gate stops the road traffic for five minutes and the trains pass the crossing twice during every hour. Let us mark on the dial of the timepiece all the positions of the minute hand corresponding to the instants when the buses pass the crossing: to this end we mark the time t_0 minutes when the first bus passes the crossing and then lay off along the circumference of the dial in the clockwise direction the arcs corresponding to time intervals of T minutes to obtain the consecutive marks indicating the instants $t_0 + T$, $t_0 + 2T$, etc. The magnitudes of t_0 and T_0 should be chosen so that none of these marks falls inside the intervals from 0 minutes to 5 minutes and from 38 minutes to 43 minutes which are shaded in Fig. 12.

First of all it should be noted that if the time table for the buses is worked out in the required manner then *among the marks* *on the dial obtained as described above there can be not more than 12 different marks* *. Indeed, let R and R_1 be two marks on the dial lying at the shortest distance from each other and let $ROR_1 = \alpha°$ be the angle corresponding to them (here O is the centre of the dial). The angle $\alpha°$ corresponds to a time interval of τ minutes and the marks R and R_1 indicate some instants $t_0 + iT$ and $t_0 + jT$ where i and j are integers (for definiteness, let $j > i$). Then the interval between the instants when the ith and the jth buses approach the crossing is equal to k hours plus or minus τ minutes where k is an integral number; for definiteness, let us assume that the interval between the buses is equal to k hours plus τ minutes so that the arc $\smile RR_1$ is laid off along the circumference of the dial in the clockwise direction as shown in Fig. 12 (in the case when this interval is equal to k hours minus τ minutes the arc $\smile RR_1$ is laid off in the opposite direction but the argument remains almost the same). In this case the $[j+(j-i)]$th bus (that is the $(2j-i)$th bus) follows the jth bus after k hours and τ minutes and to the moment this bus approaches the crossing there corresponds a mark R_2 such that $\smile R_1R_2 = \smile RR^1 = \alpha°$. Similarly, to the $[(2j-i)+1+(j-i)]$th bus (that is to the $(3j-2i)$th bus) there corresponds a mark R_3 such that $\smile R_2R_3 = \alpha^0$ etc. On the other hand, the $[i-(j-i)]$th bus (that is the $(2i-j)$th bus) approaches the crossing k hours and τ minutes earlier than the ith bus, that is the mark R_{-1} corresponding to the $[i-(j-i)]$th bus is placed so that R_{-1} and R_1 lie on different sides of mark R and $\smile R_{-1}R = \smile RR_1 = \alpha°$, and so on. In this way we obtain a network of marks such that the shortest distance between them corresponds to an arc of $\alpha°$ or, which is the same, to a time interval of τ minutes. It is clear that if $\tau < 5$ (this corresponds to an angle $\alpha° < 30°$ since to an interval of 5 minutes on the dial there corresponds a central angle of 30°) then at least one of these marks falls inside an interval corresponding to one of the arcs $\smile AB$ and $\smile CD$ shaded in Fig. 12, which means that the lifting gate stops the road traffic when the corresponding bus approaches the crossing. Therefore there must be $\tau \geqslant 5$ minutes (that is $\alpha° \geqslant 30°$), and hence the total number of the marks cannot exceed $360°/30° = 12$.

Thus, the timetable for the buses corresponds to k ($k \leqslant 12$) marks R, R_1, R_2, ... on the dial. From what has been said it fol-

* It is clear that, generally, there can even be an *infinite* number of marks on the dial: it can be proved that this is the case when the interval T is *incommensurable* with an interval of 1 hour corresponding to the turn of the hour hand through an angle of 360°; in this case the marks are placed on the dial "densely" in the sense that for any point M on the dial and for an arbitrarily small arc of magnitude ε there are marks which lie at an arc distance not exceeding ε from the point M.

lows that $\smile RR_1 = \smile R_1R_2 = \smile R_2R_3 = \ldots$, which means that $RR_1R_2 \ldots R_{k-1}$ is a *regular* k-gon inscribed in the circumference of the dial (we shall denote this k-gon as M_k). Hence, we should determine all the values of k ($k \leqslant 12$) for which regular k-gons M_k can be inscribed in the circumference of the dial so that none of the vertices of M_k falls inside the arcs $\smile AB$ and $\smile CD$ shaded in Fig. 12. It is evident that for $k = 1$ (this means that the buses follow one another with an interval of one hour) and for $k = 2$ (in this case the interval between the buses is half an hour, and it is required to indicate two marks R and R_1 on the dial not falling inside the arcs $\smile AB$ and $\smile CD$ and lying on the *opposite sides* of a diameter of the dial) it is possible to work out the timetable in the required manner. The condition $T < 30$ minutes means that it is required to find the solutions of the problem corresponding to the values of k exceeding 2. It is also evident that for $k = 12$ (that is $\alpha° = 30°$ and $\tau = 5$ minutes) this is impossible. For, in this case, there are no marks within the arc $\smile AB$ only if its end points A and B coincide with some marks R and R_1; then the 9th mark corresponds to the instant $(0 + 8 \cdot 5)$ minutes $= 40$ minutes and therefore it falls inside the arc CD. Similarly, the values $k = 10$, $k = 9$ and $k = 7$ (that is the values $\tau = 60/10 = 6$ minutes, $\tau = 60/9 = 6\frac{2}{3}$ minutes and $\tau = 60/7 = 8\frac{4}{7}$ minutes) should also be discarded. For instance, if $\tau = 6$ minutes and if R does not fall inside the arc $\smile AB$ then the mark R which is the nearest to the point A must be within the interval from 5 minutes to 6 minutes and then the mark corresponding to the 7th bus lies within the interval from $5 + 6 \cdot 6 = 41$ minutes to $6 + 6 \cdot 6 = 42$ minutes which is entirely contained in the arc $\smile CD$. (Let the reader check that if the mark R falls inside the interval from 5 minutes to $6\frac{2}{3}$ minutes or inside the interval from 5 minutes to $8\frac{4}{7}$ minutes then the 6th or the 5th bus, respectively, approaches the crossing at instants to which correspond marks falling inside the arc $\smile CD$.) Finally, the values $k = 11$, $k = 8$, $k = 6$, $k = 5$, $k = 4$ and $k = 3$ are admissible and they correspond to the values $\tau = 60/11 = 5\frac{5}{11}$ minutes, $60/5 = 12$ minutes, $60/4 = 15$ minutes and $60/3 = 20$ minutes respectively. For instance, if $k = 11$ then the buses may approach the crossing, say, at the instants 5 minutes, $10\frac{5}{11}$ minutes, $15\frac{10}{11}$ minutes, $21\frac{4}{11}$ minutes, $26\frac{9}{11}$ minutes, $32\frac{3}{11}$ minutes, $37\frac{8}{11}$ minutes, $43\frac{2}{11}$ minutes, $48\frac{7}{11}$ minutes, $54\frac{1}{11}$ minutes, and $59\frac{6}{11}$ minutes, and if $k = 5$ then the buses may approach the crossing at the instants 8 minutes, $8 + 12 = 20$ minutes, 32 minutes, 44 minutes and 56 minutes.

Thus, *under the condition* $T < 30$ *minutes the timetable for the buses can be worked out for* $T = 20$ *minutes,* $T = 15$ *minutes,* $T = 12$ *minutes,* $T = 7\frac{1}{2}$ *minutes and* $T = 5\frac{5}{11}$ *minutes and for these values of* T *only.*

119. Let the number N be written as $N = \overline{abc}$ where a, b and c are the digits of N and the bar designates the number N itself. It is clear that for the numbers $N = 100$, $N = 200$, $\ldots$, $N = 900$ we have $N/(a + b + c) = 100$. Further, if the number N does not end with two noughts then $b + c > 0$ and $a + b + c \geqslant a + 1$, and since the digit a of the number N stands in the hundreds place, we have $N < (a + 1) \cdot 100$ and

$$\frac{N}{a+b+c} < \frac{(a+1)\cdot 100}{a+1} = 100$$

Thus, *the greatest value of the ratio we consider is equal to* 100; *this greatest value is attained only for the numbers multiple of* 100.

Remark. It can similarly be proved that the greatest value of the ratio of a k-digit number $N = \overline{a_k a_{k-1} \ldots a_1}$ to the sum $a_k + a_{k-1} + \ldots + a_1$ of its digits is equal to 10^{k-1} and that this greatest value is attained only for the numbers whose last $k - 1$ digits are noughts.

120. The given number consists of 192 digits; the number obtained from the given number by deleting its 100 digits consists of 92 digits.

(a) The first digits of the number we are interested in must have the greatest possible values. We can delete 100 digits from the given 192-digit number so that the remaining digits form a number beginning with 5 nines; these 5 nines can be taken from the numbers 9, 19, 29, 39 and 49 by deleting the corresponding $8 + 19 + 19 + 19 + 19 = 84$ digits. We cannot make the next digits equal to nine because in that case to "arrive" at the nearest digit nine contained in the number 59 we should delete 19 more digits; this means that we should delete $84 + 19 = 103 > 100$ digits.

Among the remaining numbers the nearest (counting from left to right) digit 8 is contained in the number 58; in order to make this digit stand immediately after the 5 nines we should delete 17 more digits, which we are not allowed to do because $84 + 17 = 101 > 100$. Therefore the best way to achieve our aim is to delete 15 more digits preceding the digit 7 in the number 57. Then the total number of the deleted digits will be $84 + 15 = 99$, and we are allowed to delete only one more digit. It is evident that we should delete the digit 5 contained in the number 58, and thus the required number is

$$9999978596061 \dots 100$$

(b) We can make the number obtained after 86 digits have been deleted begin with 5 noughts: these noughts are contained in the numbers 10, 20, 30, 40 and 50, the number of the deleted digits being equal to $10 + 19 + 19 + 19 + 19 = 86$. We cannot make the next (the sixth) digit be equal to 0 because that nought is contained in the number 60 and to arrive at this nought we must altogether delete more than 100 digits. However, we can delete only one digit 5, after which we obtain the digit 1 following the 5 noughts which we have already obtained in the resultant number. The next digit cannot be made equal 1. However, on deleting one more digit 5 we can make the next digit of the resultant number be equal to 2. This argument can be continued and we see that on deleting $86 + 1 + 1 + 1 + 1 = 90$ digits we arrive at a number beginning with the combination 000001234 which is followed by the digits 55565758596061 ... . It is evident that the digit following the first nine digits 000001234 of the sought-for number can be equal to five but cannot be less than 5; for this digit to be equal to 5 no additional digits should be deleted. Finally, after the operations we have performed we can arrive at 0 by deleting the 10 digits preceding this digit 0, which we are allowed to do. Therefore the number we are interested in is

$$00000123450616263 \dots 100$$

This number has five noughts at the beginning; on discarding these noughts we obtain the required 87-digit number.

121. (a) The first digits of the three sought-for numbers must have the least possible values; consequently in the decimal notation these numbers have the form

$$\overline{1Aa}, \quad \overline{2Bb} \text{ and } \overline{3Cc}$$

where the symbol $\overline{xyz}$ designates the number written with the aid of the digits x, y and z.

Let us prove the following three facts: (1) $A < B < C$; (2) $a < b < c$ and (3) each of the digits a, b and c is greater than each of the digits A, B and C.

(1) If, for instance, we had $A > B$ then we should have $\overline{Aa} > \overline{Bb}$ and $\overline{1Aa}\cdot\overline{2Bb} - \overline{2Aa}\cdot\overline{1Bb} = (100 + \overline{Aa})\cdot(200 + \overline{Bb}) - (200 + \overline{Aa})\cdot(100 + \overline{Bb}) = 100(\overline{Aa} - \overline{Bb}) > 0$ and, consequently, we should have $\overline{1Bb}\cdot\overline{2Aa}\cdot\overline{3Cc} < \overline{1Aa}\cdot\overline{2Bb}\cdot\overline{3Cc}$, which is impossible.

(2) If, for instance, we had $a > b$, then we should have $\overline{1Aa} \cdot \overline{2Bb} - \overline{1Ab} \cdot \overline{2Ba} = (10 \cdot \overline{1A} + a)(10 \cdot \overline{2B} + b) -$

$$- (10 \cdot \overline{1A} + b)(10 \cdot \overline{2B} + a) = (10 \cdot \overline{2B} - 10 \cdot \overline{1A})(a - b) > 0$$

whence it should follow that $\overline{1Ab} \cdot \overline{2Ba} \cdot \overline{3Cc} < \overline{1Aa} \cdot \overline{2Bb} \cdot \overline{3Cc}$.

(3) If there were $C > a$, that is $C = a + x$ where $x > 0$ (by virtue of (1) and (2), the digit C is the greatest among A, B and C while the digit a is the smallest of the digits a, b and c) then we should have

$$\overline{1Aa} \cdot \overline{3Cc} - \overline{1Ac} \cdot \overline{3ac} = \overline{1Aa} \cdot (\overline{3ac} + 10x) - (\overline{1Aa} + x)(\overline{3ac}) =$$

$$= x(10 \cdot \overline{1Aa} - \overline{3ac}) > 0$$

whence it would follow that $\overline{1AC} \cdot \overline{2Bb} \cdot \overline{3ac} < \overline{1Aa} \cdot \overline{2Bb} \cdot \overline{3Cc}$.

From (1), (2) and (3) it follows that

$$A < B < C < a < b < c$$

and hence the sought-for product is of the form

$$147 \cdot 258 \cdot 369$$

(b) The sought-for product must have the form

$$\overline{9Aa} \cdot \overline{8Bb} \cdot \overline{7Cc}$$

By analogy with the solution of Problem 121 (a), we can prove that (1) $A < B < C$, (2) $a < b < c$ and (3) each of the digits a, b and c is less than each of the digits A, B and C.

From (1), (2) and (3) it follows that

$$a < b < c < A < B < C$$

Consequently, the sought-for product has the form

$$941 \cdot 852 \cdot 763$$

122. By the condition of the problem, we have $m + (m + 1) + \ldots + (m + k) = 1000$. According to the formula for the sum of the terms of an arithmetic progression, we have

$$\frac{2m + k}{2}(k + 1) = 1000$$

that is

$$(2m + k)(k + 1) = 2000$$

Since the number

$$(2m + k) - (k + 1) = 2m - 1$$

is odd, one of the factors in the above equality is even and the other is odd. Besides, we obviously have $2m + k > k + 1$. We see that the problem has the following solutions:

$$2m+k=2000,\quad k+1=1,\quad m=1000,\quad k=0$$
$$2m+k=400,\quad k+1=5,\quad m=198,\quad k=4$$
$$2m+k=80,\quad k+1=25,\quad m=28,\quad k=24$$

and

$$2m+k=125,\quad k+1=16,\quad m=55,\quad k=15$$

123. **(a)** Let the number N be different from any power of 2. Then we have the equality

$$N=2^k(2l+1)$$

where 2^k is the highest power of 2 by which N is divisible (the number k can be equal to zero) and $2l+1$ is the greatest odd divisor of the number N. Further, we have

$$(2^k-l)+(2^k-l+1)+\ \dots\ +(2^k-l+2l-1)+(2^k-l+2l)=$$
$$=\frac{(2l+1)(2^k-l+2^k-l+2l)}{2}=2^k(2l+1)=N$$

If several of the first of these $(2l+1)$ consecutive integers are negative (that is if $l>2^k$), then they and the corresponding first several positive numbers mutually cancel, and N can again be represented as a sum of a number (smaller than $2l+1$) of *positive* integers.

Now let us suppose that a number of the form 2^k can be represented as a sum of m consecutive positive integers n, $n+1, \dots$ $\dots, n+m-2, n+m-1$.
Then

$$2^{k+1}=2[n+(n+1)+\ \dots\ +(n+m-2)+(n+m-1)]=$$
$$=m(n+n+m-1)=m(2n+m-1)$$

The difference $(2n+m-1)-m=2n-1$ being odd, one of the numbers m and $2n+m-1$ is odd (and both numbers are different from 1 because $m>1$ and $n>0$). Consequently the last equality cannot hold since 2^{k+1} has no odd divisors different from 1.

(b) We have

$$(2n+1)+(2n+3)+(2n+5)+\ \dots\ +(2m-1)=$$
$$=\frac{(2n+1)+(2m-1)}{2}\cdot(m-n)=(m+n)\cdot(m-n)$$

Therefore if N is a number which can be represented as a sum of consecutive odd numbers it must be composite (because it can be represented as a product of two factors $m+n$ and $m-n$). On the other hand, every odd composite number N can be written in the form of a product of two *odd* factors a and b $(a \geqslant b)$, and consequently we have $N = ab = (m+n)(m-n)$ where $m = (a+b)/2$ and $n = (a-b)/2$; this means that N is equal to the sum of the odd numbers from $a-b+1$ to $a+b-1$.

Further, the factors $m+n$ and $m-n$ in the formula $N = (m+n)(m-n)$ are simultaneously even or odd; if the number N is even, these factors must obviously be even, and in this case N is divisible by 4 (and both $m+n$ and $m-n$ are divisible by 2). Consequently if an even number N is not divisible by 4 it cannot be represented as a sum of consecutive odd numbers. In case N is of the form $N = 4n$ (that is N is divisible by 4), the number N can be represented as the sum of the two consecutive odd numbers $2n-1$ and $2n+1$.

(c) It is clear that

$$(n^{k-1}-n+1)+(n^{k-1}-n+3)+\dots+(n^{k-1}-1)+$$
$$+(n^{k-1}+1)+\dots+(n^{k-1}+n-3)+(n^{k-1}+n-1)=$$
$$=\frac{(n^{k-1}-n+1)+(n^{k-1}+n-1)}{2}\cdot n=n^k$$

(all the summands in this sum are odd since n^{k-1} and n are simultaneously even or odd).

124. Let us denote four consecutive numbers as n, $n+1$, $n+2$ and $n+3$. The sum of their product plus unity can be written in the form

$$n(n+1)(n+2)(n+3)+1=[n(n+3)][(n+1)(n+2)]+1=$$
$$=(n^2+3n)(n^2+3n+2)+1=(n^2+3n)^2+2(n^2+3n)+1=$$
$$=(n^2+3n+1)^2$$

and consequently this sum is the square of the whole number (n^2+3n+1).

125. Let us prove that these numbers assume not more than four different values. Suppose that, on the contrary, there are five numbers a_1, a_2, a_3, a_4 and a_5 among the given $4n$ numbers which are pairwise different. Let us suppose that $a_1 < a_2 < a_3 < a_4 < a_5$.

We shall start with considering the numbers a_1, a_2, a_3 and a_4. By the hypothesis, they can be arranged as a geometric progression. Therefore the product of two of them (which are the extremes of the proportion) is equal to the product of the other two numbers (the means of the proportion). But this is only possible when

$$a_1a_4 = a_2a_3$$

(the equality $a_1a_3 = a_2a_4$ is impossible because $a_1 < a_2$ and $a_3 < a_4$; it is evident that the equality $a_1a_2 = a_3a_4$ cannot hold either).

Now let us consider the numbers a_1, a_2, a_3 and a_5. In just the same manner we can prove that $a_1a_5 = a_2a_4$. Consequently, $a_1a_4 = a_1a_5$ whence it follows that $a_4 = a_5$, which contradicts the hypothesis.

We have thus proved that each of the $4n$ numbers assumes one of not more than four different values. Therefore not less than n numbers among the $4n$ given numbers assume one of these values.

126. Let us take nine weights weighing n^2, $(n+1)^2$, $(n+2)^2, \ldots$ $\ldots$, $(n+8)^2$ respectively and divide them into the following three groups:

1st group: n^2, $(n+5)^2$, $(n+7)^2$:

$$n^2+(n+5)^2+(n+7)^2=3n^2+24n+74;$$

2nd group: $(n+1)^2$, $(n+3)^2$, $(n+8)^2$:

$$(n+1)^2+(n+3)^2+(n+8)^2=3n^2+24n+74;$$

3rd group: $(n+2)^2$, $(n+4)^2$, $(n+6)^2$:

$$(n+2)^2+(n+4)^2+(n+6)^2=3n^2+24n+56$$

We see that the total weight of the first group and the total weight of the second group are equal and that the third group is lighter by 18 than each of the former groups. Next we take nine weights weighing $(n+9)^2$, $(n+10)^2$, $(n+11)^2$, $\ldots$, $(n+17)^2$ and divide them in a similar manner into three groups so that the first and the third groups are of the same weight while the second group is lighter by 18 than each of the former groups. Finally, let us take nine weights weighing $(n+18)^2$, $(n+19)^2$, $(n+20)^2, \ldots$ $\ldots$, $(n+26)^2$ respectively and divide them into three groups so that the second and the third groups are of one weight while the first group is lighter by 18 than each of them. Next, on combining the first groups, the second groups and the third groups which we have formed we see that any 27 weights weighing n^2, $(n+1)^2$, $(n+2)^2$, $\ldots$, $(n+26)^2$ respectively can be divided into three groups of equal weight for any $n = 0, 1, 2, 3, \ldots$.

127. First of all we note that the conditions of the problem imply that *all the weights simultaneously weigh an even or an odd number of grams.* Indeed, since any 12 of the weights can be divided into two groups of the same weight, it follows that the total weight of any group of 12 weights is expressed by an even number. Besides, the weight of a group of 12 weights remains even when any of them is replaced by the remaining 13th weight, which is only possible when the weight of each of the 12 weights and the weight of the remaining 13th weight are simultaneously even or odd, whence follows what was said above.

Now let us consider a new set of weights weighing the number of grams the former weights weigh minus the number of grams the lightest of them weighs (or minus the sum of the weights of the lightest of them in case there are several such weights). It is evident that the new set of weights also satisfies the conditions of the problem, and consequently, the numbers of grams they weigh are simultaneously even or odd. Among the new weights there are some whose "weight" is equal to zero, and hence the weights in the new set must be *even*. Now let us pass to a third set of weights which weighs the number of grams the second weights weigh divided by 2. The third set of weights also satisfies the conditions of the problem.

Next let us suppose that not all of the original weights weigh an equal number of grams. Then not all the weights in the second set are equal to zero. Therefore we can continue the process of the consecutive division of the number of grams all the weights weigh by two and arrive eventually at a set of weights some of which weigh an even number of grams (for instance, they can be of "zero weight") while the other weigh an odd number of grams. However, as has been shown, there exists no such set of weights satisfying the condition of the problem. The contradiction we have arrived at proves the assertion of the problem.

Remark. In the condition of the problem it is required that the weights should be expressed by *integral* numbers. However, it can easily be seen that if they are expressed by *rational* numbers instead of integers the result remains the same. Indeed, on multiplying all the weights by the common denominator of the rational numbers we reduce the problem to the case of *integral* weights. Moreover, in case the weights are expressed by *irrational* numbers we can also prove that they are equal to one another using the fact that it is possible to find rational numbers which are arbitrarily close to the given irrational numbers (let the reader carry out the proof for this general case; by the way, the rigorous proof is rather intricate).

128. First of all we note that if two four-tuples a_k, b_k, c_k, d_k and a_l, b_l, c_l, d_l obtained for some natural numbers k and l $(k \neq l)$ in the described manner coincide then *either all the numbers* $a_k = a_l$, $b_k = b_l$, $c_k = c_l$ and $d_k = d_l$ *are equal to zero or they all are positive.* Indeed, if at least one of the numbers a, b, c and d is equal to *zero* then at least at the fourth "step" we arrive at a four-tuple consisting of zeros which then repeats indefinitely, and there are not 2 four-tuples among the preceding four-tuples which coincide. For, if there is exactly *one* number equal to zero, say $a = 0$, then the first 4 four-tuples are of the form

$$0,\ b,\ c,\ d; \quad 0,\ bc,\ cd,\ 0; \quad 0,\ bc^2d,\ 0,\ 0; \quad 0,\ 0,\ 0,\ 0$$

and there are not 2 four-tuples among them which coincide because zeros contained in them occupy different places. It is also clear that if *more than one* number among a, b, c, d is equal to zero then we also arrive at a four-tuple of zeros not later than at the 4th step, and it can again be easily checked that the preceding four-tuples not all of whose members are zeros must necessarily be pairwise different.

What has been said exhausts the investigation of the case when $abcd = 0$. Now let us suppose that $abcd \neq 0$. It is obvious that in this case all the numbers contained in all four-tuples are different from zero. Further, if among the numbers a, b, c, d there is only one negative number, say the first one, then denoting the positive numbers by the symbol "+" and the negative numbers by the symbol "—" we can describe the alternation of the signs in the first five four-tuples with the aid of the following scheme:

$$-+++;\quad -++-;\quad -+-+;\quad ----;\quad ++++$$

Thus, here the 5th four-tuple consists only of positive numbers, and there are not two four-tuples among the preceding ones which coincide because the alternation of the signs in these four-tuples is different. This scheme also shows that if among the numbers a, b, c, d there are two negative numbers standing side by side (for instance, this is the case for the 2nd of the above four-tuples because the numbers in the four-tuples are considered as being arranged in a "cyclic" order and therefore the 1st and the 4th numbers should be regarded as "standing side by side") or if there are two negative numbers not adjoining each other (see the 3rd four-tuple) or four negative numbers (for instance, see the 4th four-tuple) then we again arrive at a four-tuple consisting only of positive numbers not later than at the 4th step (and the preceding four-tuples are all different). The case of a four-tuple containing three negative numbers and one positive number is considered in a similar way: such a four-tuple is transformed into the 2nd of the above four-tuples immediately after the first step:

$$+---;\quad -++-;\quad -+-+;\quad ----;\quad ++++$$

Thus, in our further argument we can assume that *all the numbers a, b, c, d are positive.* Let us put $abcd = p$; we obviously have

$$a_1b_1c_1d_1 = (ab)(bc)(cd)(da) = (abcd)^2 = p^2$$

and, similarly, $a_2b_2c_2d_2 = (a_1b_1c_1d_1)^2 = p^4$; $a_3b_3c_3d_3 = p^8$; generally we can write $a_kb_kc_kd_k = (p)^{2^k}$ where $k = 0, 1, 2, \ldots$ (here a_0, b_0, c_0, d_0 designate the original numbers a, b, c, d respectively). Therefore if the four-tuples a^k, b^k, c^k, d^k and a_l, b_l, c_l, d_l (where $l > k$) coincide then

$$p^{2^k} = a_kb_kc_kd_k = a_lb_lc_ld_l = p^{2^l}$$

$$\text{and hence} \quad p^{2^l-2^k} = 1, \quad \text{that is} \quad p = 1$$

Now let us suppose that $abcd = 1$ (it is clear that it is sufficient to investigate this case only). Then, since $cd = 1/ab$, one of the two numbers ab and cd is not less than 1 and the other is not greater than 1. For definiteness, let us suppose that $ab=\alpha\geqslant 1$ and $cd = 1/\alpha \leqslant 1$. Similarly, among the numbers bc and $da = 1/bc$ one is not less and the other is not greater than 1. For definiteness, let us suppose that $bc = \beta \geqslant 1$ and $da = 1/\beta \leqslant 1$ and that $\alpha \geqslant \beta$ (all the other possible cases do not essentially differ from the one under consideration). In this case we obtain in succession the following number four-tuples:

$$a,\ b,\ c,\ d;\quad \alpha,\ \beta,\ \frac{1}{\alpha},\ \frac{1}{\beta};\quad \alpha\beta,\ \frac{\beta}{\alpha},\ \frac{1}{\alpha\beta},\ \frac{\alpha}{\beta};\quad \beta^2,\ \frac{1}{\alpha^2},\ \frac{1}{\beta^2},\ \alpha^2$$

Here the greatest number in the 2nd four-tuple is equal to α, the greatest number in the 3rd four-tuple is equal to $\alpha\beta$ ($\alpha\beta \geqslant \alpha$), the greatest number in the 4th four-tuple is equal to α^2 ($\alpha^2 \geqslant \alpha\beta$) etc. (it should be noted that the only difference between the 4th four-tuple and the 2nd four-tuple is that in the former the role of α and β is played by the numbers α^2 and β^2 respectively).

Thus, we see that *the greatest number belonging to a four-tuple does not decrease when the operation of forming new four-tuples is performed repeatedly.* If $\alpha > 1$ and $\beta > 1$ then this greatest number even permanently *increases*, and therefore in this case there are not two four-tuples each of which is different from the 1st one that coincide with each other. Moreover, if at least one of the two numbers α and β is different from 1, then in this case as well there are not two four-tuples different from the 1st one which coincide. Indeed, if $\alpha > 1$ and $\beta = 1$ then the 2nd four-tuple and the following four-tuples have the form

$$\alpha,\ 1,\ \frac{1}{\alpha},\ 1;\quad \alpha,\ \frac{1}{\alpha},\ \frac{1}{\alpha},\ \alpha;\quad 1,\ \frac{1}{\alpha^2},\ 1,\ \alpha^2$$

Thus, in this case the greatest number belonging to a four-tuple increases after two operations have been performed and therefore the resultant four-tuple cannot coincide with any of the preceding four-tuples, and the numbers contained in the first two four-tuples are also different. Consequently, if the kth and the lth four-tuples coincide (where $l > k > 1$) then $\alpha = \beta = 1$; in this case *all the four-tuples coincide with one another beginning with the 2nd one* (these four-tuples consist of ones only).

Up till now we have not considered the 1st four-tuple a, b, c, d in our argument; it is also necessary to find whether this four-tuple can coincide with one of the following four-tuples. It turns out that such a coincidence is impossible because, as has been shown, all the four-tuples, beginning with the 2nd one, contain pairs of numbers whose products are equal to 1, and therefore if the four-tuple a, b, c, d coincides with one of the following four-tuples and, say, $bc = \beta > 1$ and $ab = \alpha \geqslant \beta > 1$ then, since $cd = \frac{1}{\alpha} < 1$ and $da = \frac{1}{\beta} < 1$, we must assume that $ac = bd = 1$. This assumption leads to the following values of the numbers belonging to the first four-tuple:

$$\sqrt{\frac{\alpha}{\beta}}, \quad \sqrt{\alpha\beta}, \quad \sqrt{\frac{\beta}{\alpha}}, \quad \frac{1}{\sqrt{\alpha\beta}}$$

(why?), which again allows us to compare the greatest numbers α and $\sqrt{\alpha\beta}$ belonging to the 1st and to the 2nd four-tuple respectively. These greatest numbers coincide only when $\beta = \alpha$, that is only when the 1st four-tuple coincides with the 2nd four-tuple corresponding to the case $\beta = 1$ and when this 1st four-tuple undergoes the further transformation analogous to that of the 2nd four-tuple. The case $\beta = 1$ is investigated analogously.

This consideration concludes the proof of the theorem.

Remark. The solution of the problem also allows us to estimate the number of operations which are necessary for all resulting four-tuples of number to become coincident with one another (in the case when not all four-tuples are different). As we see, in the case when $abcd = 0$ all the four-tuples coincide beginning with the 4th one, in the case when the numbers a, b, c, d are positive and satisfy the conditions $ab = bc = cd = da = 1$ all four-tuples coincide beginning with 2nd one (and if these equalities are not fulfilled there are not two four-tuples coinciding with each other) as in the case when the numbers are not necessarily positive four more steps may be needed in order to transform all the numbers into positive ones.

129. Since the square of each of the numbers a_i (where $i = 1, 2, \ldots, N = 2^k$) is equal to 1, the first three number sequences have the following form:

$$\begin{array}{lllll} a_1; & a_2; & \ldots; & a_{N-1}; & a_N \\ a_1a_2; & a_2a_3; & \ldots; & a_{N-1}a_N; & a_Na_1 \\ a_1a_2^2a_3 = a_1a_3; & a_2a_3^2a_4 = a_2a_4; & \ldots; & a_{N-1}a_N^2a_1 = a_{N-1}a_1; & a_Na_1^2a_2 = a_Na_2 \end{array}$$

Thus, every number belonging to the 3rd sequence is equal to the product of the corresponding number belonging to the first sequence by another member of that sequence whose serial index exceeds the index of the former by 2 (the numbers in the sequence $a_1, a_2, \ldots, a_N$ are regarded as being ordered in a cyclic way, that is the number a_N is followed by the number a_1 after which the number a_2 follows again and so on). Similarly, after two more steps we arrive at a sequence which is obtained from the 3rd one in the same way as the 3rd sequence is obtained from the 1st one, that is we obtain the sequence

$$(a_1a_3)(a_3a_5)=a_1a_5;\quad (a_2a_4)(a_4a_6)=a_2a_6;\quad \ldots;\quad (a_Na_2)(a_2a_4)=a_Na_4$$

Every number belonging to the last sequence is obtained from the corresponding number belonging to the 1st sequence (ordered in the cyclic manner) by multiplying it by the number whose serial index exceeds that of the former by 4. After 4 more steps we arrive at a sequence which is obtained from the 5th one in just the same manner as the 5th sequence is obtained from the 1st one, that is we obtain the sequence

$$(a_1a_5)(a_5a_9)=a_1a_9;\quad (a_2a_6)(a_6a_{10})=a_2a_{10};\quad \ldots;\quad (a_Na_4)(a_4a_8)=a_Na_8$$

whose every member is obtained from the corresponding number of the original sequence by multiplying it by the number whose serial index exceeds that of the former by 8.

Generally, after 2^p steps we arrive at a sequence of the form

$$a_1a_{2^p+1};\quad a_2a_{2^p+2};\quad \ldots;\quad a_Na_{2^p}$$

whose every member is obtained by multiplying the corresponding number belonging to the original sequence by the number whose index exceeds by 2^p that of the former. It follows that after 2^k steps we arrive at a sequence obtained from the original sequence (regarded as being ordered in the cyclic manner) by means of the pairwise multiplication of the numbers belonging to that sequence whose indices differ by $2^k=N$, that is we obtain the sequence

$$a_1a_1=a_1^2=1;\quad a_2a_2=a_2^2=1;\quad \ldots;\quad a_Na_N=a_N^2=1$$

consisting of ones only.

130. First of all let us show that when we pass from the original number sequence $a_1, a_2, \ldots, a_n$ to the "derived" sequence $a_1', a_2', \ldots, a_n'$, the differences between the numbers are "smoothed" in the sense that under this transformation from one sequence to the other *the difference between the greatest and the smallest numbers does not increase.* Indeed, since half the sum of two numbers (that is their arithmetic mean) is always not greater than the greatest of them (it is equal to the greatest number only in the case when these numbers coincide), the greatest of the numbers $a_1', a_2', \ldots, a_n'$ which is equal to half the sum of some two numbers belonging to the original sequence does not exceed the greatest of these two numbers and therefore it does not exceed the greatest of all n numbers a_i. Hence, *when we pass from the sequence a_i to the sequence a_i' the greatest number can only decrease.* Moreover, this argument shows that the greatest of the numbers a_i' can be *equal* to the greatest of the numbers a_i only in the case when the greatest number A among the numbers a_i is repeated in the sequence a_i several times and when in this sequence (which is considered as being ordered in a cyclic manner so that the number a_n is again followed by a_1) there are two *neighbouring* numbers equal to A. Further, it can easily be seen that if the longest chain of numbers equal to A contained in the sequence a_i is of length k (where $k < n$), then the sequence a_i' contains a chain of $k-1$ numbers equal to A which follow one another. For instance, if $a_{i+1} = a_{i+2} = \ldots = a_{i+k} = A$ (while $a_i < A$ and $a_{i+k+1} < A$) then $a'_{i+1} = a'_{i+2} = \ldots = a'_{i+k-1} = A$ (while $a_i' < A$ and $a'_{i+k} < A$). Therefore after $k-1$ steps we arrive at the numbers $a_1^{(k-1)}, \ldots, a_n^{(k-1)}$ among which there are not two numbers equal to A that stand side by side, and at the next (the kth) step the greatest of the numbers under consideration *decreases*. Consequently, *if not all the numbers a_i are equal to one another then after the* $(n-1)$th *repetition of the procedure described in the condition of the problem the greatest of the numbers under consideration must decrease.* In just the same way it can be proved that *the smallest among the numbers under consideration can never decrease*, and if not all the numbers are equal to one another then after the $(n-1)$th repetition of the procedure the smallest number must *increase*.

If the numbers $a_1, a_2, \ldots, a_n$ and all the following numbers obtained from them in succession are integers then their "smoothing", that is the decreasing of the difference between the greatest and the smallest of them, must eventually lead to the case when this difference becomes equal to zero, which means that all the numbers become equal to one another. Indeed, the original difference $A - a = \max_i a_i - \min_i a_i$ is equal to a positive integer p; when $A = \max_i a_i$ decreases or when $a = \min_i a_i$ increases this difference decreases by not less than unity, and consequently, after not more than p such steps it must become equal to zero. Thus, the assertion of the problem will be proved if we show that *in the case when the original numbers $a_1, a_2, \ldots, a_n$ are not all equal to one another we can never arrive at a sequence of equal numbers.*

Now let us study in which way equal numbers $b_1 = a_1^{(m)}$, $b_2 = a_2^{(m)}, \ldots, b_n = a_n^{(m)}$ can be obtained from the numbers

$$c_1 = a_1^{(m-1)}, \quad c_2 = a_2^{(m-1)} \quad \ldots, \quad c_n = a_n^{(m-1)}$$

among which not all are equal.

It is clear that to this end it is necessary that the numbers with odd indices belonging to the sequence $c_1, c_2, c_3, \ldots$ should coincide with one another and the numbers with even indices should coincide with one another, that is it is necessary that the equalities

$$c_1 = c_3 = c_5 = \ldots = c \quad \text{and} \quad c_2 = c_4 = c_6 = \ldots = C \qquad (*)$$

should be fulfilled. However, since the numbers $c_1, c_2, c_3, \ldots$ are ordered in a cyclic manner, that is the number c_{n+1} should be considered to be coincident with c_1, equalities (*) (where $C \neq c$) cannot hold when $n = 2l + 1$ is an *odd* number (that is when the number $n + 1 = 2l + 2$ is even). Therefore it only remains to suppose that the number n is *even*: $n = 2l$. Now let us make one more (backward step), that is let us consider the numbers $d_1 = a_1^{(m-2)}$, $d_2 = a_2^{(m-2)}, \ldots, d_n = a_n^{(m-2)}$ preceding the numbers $c_1, c_2, \ldots, c_n$. We obviously have

$$c_1 = \frac{d_1 + d_2}{2}, \quad c_3 = \frac{d_3 + d_4}{2}, \ldots, \quad c_{2l-1} = \frac{d_{2l-1} + d_{2l}}{2} \qquad (**)$$

and

$$c_2 = \frac{d_2 + d_3}{2}, \quad c_4 = \frac{d_4 + d_5}{2}, \ldots, \quad c_{2l} = \frac{d_{2l} + d_1}{2} \qquad (***)$$

Equalities (*) and (**) imply that $d_1 + d_2 + d_3 + \ldots + d_{2l} = 2lc$ and equalities (*) and (***) imply that $d_1 + d_2 + d_3 + \ldots + d_{2l} = 2lC \neq 2lc$. We have thus arrived at a contradiction, which completes the proof.

131. First of all let us find for what numbers x, y and z the coincidence of the triples (x_n, y_n, z_n) and (x, y, z) is possible. Since all the triples, beginning with (x_1, y_1, z_1), must be nonnegative the numbers x, y and z must themselves be *nonnegative*. Further, let us assume that $x \geqslant y \geqslant z$ and $x_i \geqslant y_i \geqslant z_i$ for all $i = 1, 2, 3, \ldots$. Since we obviously have $x_i = x_{i-1} - z_{i-1}$ for all $i \geqslant 1$ (where by x_0 and z_0 are meant the numbers x and z respectively), there must be $x \geqslant x_1 \geqslant x_2 \geqslant x_3, \ldots$, and if at least one of the numbers z_i $(i = 0, 1, 2, \ldots)$ is positive then $x_{i+1} < x_i < x$ (and also $x_j < x$ for all $j > i$). Therefore if $x_n = x$, then $z_i = 0$ for $i = 0, 1, \ldots, n - 1$. Thus, we must have $z = 0$ and $z_1 = 0$, whence it follows that either $y = x$ (and then $z_1 = x - y = 0$) or $y = z = 0$ (and then $z_1 = y - z = 0$). Hence, we see that for the triple (x, y, z) to coincide with some triple (x_n, y_n, z_n) it is *necessary* that (for $x = 1$) the original triple should have the form (1, 1, 0) or (1, 0, 0). The second case can be immediately discarded because from the triple (1, 0, 0) we pass to the triple (1, 1, 0) distinct from the original triple. Therefore if $x = 1$ and the triple (x, y, z) coincides with (x_n, y_n, z_n), then (x, y, z) coincides with the triple (1, 1, 0) (and the triples (x_n, y_n, z_n) are of the same structure *for all* n).

132. (a) It should be noted that a difference of two numbers is even or odd depending solely on whether the minuend and the subtrahend are even or odd. Let us agree to symbolize an even number by the letter e and an odd number by the letter o. Using this notation we can indicate (symbolically) the following six essentially different combinations of the original numbers A, B, C, D: 1° — e, e, e, e; 2° — e, e, e, o; 3° — e, e, o, o; 4° — e, o, e, o; 5° — e, o, o, o; 6° — o, o, o, o; all the other combinations can be obtained from these six combinations with the aid of cyclic permutations of the numbers (that is by permutations which do not change the order of the numbers; here the 1st number is considered as following the 4th one). Let us show that in all the cases *we need not more than four steps to pass to a four-tuple of even numbers*. Indeed, combination 1° itself consists of four even numbers; from combination 6° we pass to combination 1° after one step; from combination 4° we pass to combination 6° after one step and, consequently, to obtain combination 1° from 4° we need two steps; from combination 3° we pass to combination 4° on making one step and therefore we need three steps to arrive at combination 1°; finally, from combinations 2° and 5° we arrive at combination 3° on making one step (in the case of combination 5° we arrive at a combination obtained from 3° with the aid of a cyclic permutation) and hence four steps are needed to obtain combination 1°. Thus, in all the cases it is sufficient to perform four operations to arrive at a four-tuple of even numbers.

Now let us continue the process of forming new four-tuples. As before, we easily show that on making four more steps we arrive at numbers divisible by 4 and that on making again four additional steps we obtain numbers divisible by 8 and so on. Thus, on continuing this process sufficiently long we can arrive at a four-tuple of numbers divisible by *any* preassigned power of two with an arbitrarily large exponent. On the other hand, since the absolute values of the numbers do not increase, this means that eventually we must arrive at a four-tuple consisting of zeros only (if all the numbers A, B, C, D are less than 2^n then it is clear that we must arrive to a four-tuple of zeros on making $4n$ steps or, perhaps, a smaller number of steps).

Remark. We can similarly show that if there are 8 or 16 ... or any other number of the form $m = 2^k$ (different from 4) of positive integers then, performing operations analogous to the above, we arrive after a finite number of steps at m numbers equal to zero. If m is not equal to a power of two, the situation can be different; for instance, starting with the triple of numbers 1, 1, 0 we can never arrive at the triple 0, 0, 0:

$$\begin{matrix} 1, & 1, & 0 \\ 0, & 1, & 1 \\ 1, & 0, & 1 \\ 1, & 1, & 0 \end{matrix}$$

(cf. the solution of Problem 131).

(b) It is clear that if the numbers $A = a_1/a_0$, $B = b_1/b_0$, $C = c_1/c_0$ and $D = d_1/d_0$ are *rational fractions* then, on multiplying them by a factor k (for instance, by the common denominator $a_0b_0c_0d_0$ of all the fractions) we obtain *integers* $A' = kA$, $B' = kB$, $C' = kC$ and $D' = kD$, and hence from the fact that the assertion of Problem 132 (a) is true for the numbers A', B', C' and D' it follows that it is also true for the numbers A, B, C and D.

In case A, B, C, and D are *irrational numbers* the assertion of Problem 131 (a) *may be false*. To prove what has been said it suffices to indicate at least one four-tuple of numbers, A, B, C, D for which it is false. Let us put $A = 1$, $B=x$, $C=x^2$ and $D=x^3$ where x is a positive number which can be chosen arbitrarily. Then we obviously have

$$A_1 = |x-1|,$$
$$B_1 = |x^2 - x| = x|x-1|,$$
$$C_1 = |x^3 - x^2| = x^2|x-1|,$$
$$D_1 = |x^3 - 1| = (x^2+x+1)|x-1|$$

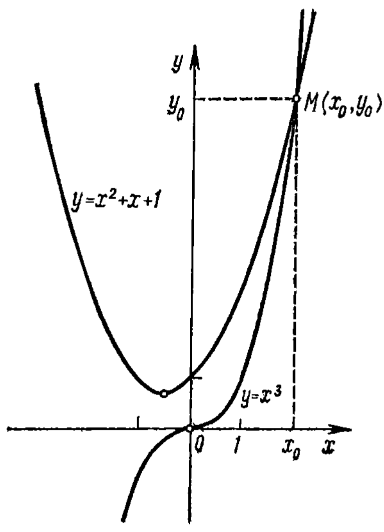


Fig. 13

that is the numbers A_1, B_1, C_1 and D_1 are proportional to 1, x, x^2 and x^2+x+1 respectively. Therefore if we manage to choose x so that the equality

$$x^2 + x + 1 = x^3 \qquad (*)$$

is fulfilled then the numbers A_1, B_1, C_1 and D_1 will be proportional to A, B, C and D and the process of forming new consecutive fou -tuples of numbers will last indefinitely. As is seen from Fig. 13 where the graphs of the functions

$$y = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

and $y = x^3$ are shown (the former graph is a (quadratic) *parabola* and the latter is a *cubic parabola*), the corresponding curves intersect at a point $M(x_0, y_0)$. Hence, $x = x_0$ is a (positive) root of equation (*). We have thus shown that the numbers $A = 1$, $B = x_0$, $C = x_0^2$ and $D = x_0^3 = x_0^2 + x_0 + 1$ (which are obviously irrational) are such that the assertion of Problem 132 (a) is false for them.

133. (a) It can easily be seen that the following array of the first 100 whole numbers satisfies the condition of the problem:

10	9	8	7	6	5	4	3	2	1	20	19	18	17	16	15	14	13	12	11
30	29	28	27	26	25	24	23	22	21	40	39	38	37	36	35	34	33	32	31
50	49	48	47	46	45	44	43	42	41	60	59	58	57	56	55	54	53	52	51
70	69	68	67	66	65	64	63	62	61	80	79	78	77	76	75	74	73	72	71
90	89	88	87	86	85	84	83	82	81	100	99	98	97	96	95	94	93	92	91

(b) Let $a_1^{(1)}$ be the first (the leftmost) of the numbers written as a sequence consisting of the 101 numbers from 1 to 101; let $a_2^{(1)}$ be the first among the other numbers in the sequence which exceeds $a_1^{(1)}$; let $a_3^{(1)}$ be the first of the numbers following $a_2^{(1)}$ which exceeds $a_2^{(1)}$ and so on. In this way we obtain the increasing number sequence $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_{i_1}^{(1)}$. If there are more than 10 numbers in this sequence (that is if $i_1 > 10$) then we obtain a solution of the problem. In case $i_1 \leqslant 10$ we delete all the numbers $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_{i_1}^{(1)}$ and choose a new increasing number sequence $a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots, a_{i_2}^{(2)}$ from the remaining $101 - i_1$ numbers by performing *just the same operation*. On continuing this process we select from the given 101 numbers a set of increasing sequences. If at least one of these sequences contains more than 10 numbers we obtain a solution of the problem. Hence it only remains to consider the case when none of the sequences we have selected contains more than 10 numbers.

Since the total number of the given integers is equal to 101, in the general case the total number k of the increasing number sequences we have selected cannot be less than 11. In the case under consideration we can assert that from the given 101 numbers it is possible to choose 11 numbers arranged in a *decreasing* order. These numbers are chosen beginning with the end of the sequence in the following manner. Let the last number of that sequence be equal to the last number $a_{i_k}^{(k)}$ of the last of the above increasing sequences. Next we choose from the last but one sequence a number which is the closest to $a_{i_k}^{(k)}$ from the left. This number exceeds $a_{i_k}^{(k)}$ because, if otherwise, then in the process of constructing the last but one sequence we should write the number $a_{i_k}^{(k)}$ after that number whereas in reality the number $a_{i_k}^{(k)}$ belongs to another sequence. In just the same way we then take a number belonging to the third (counting from the end) sequence which lies on the left of the number belonging to the last but one sequence and is the closest to it, etc. In this way we construct a number sequence which increases if we read the numbers from right to left, that is we obtain a decreasing sequence (counting, as usual, from right to left); the number of the terms of this sequence is equal to the number k of the increasing sequences selected before, and hence this number is not less than 11.

Remark. We can prove completely similarly that, given $(n-1)^2$ positive integers, it is possible to arrange them so that in the resultant sequence there are not n numbers forming an increasing subsequence or a decreasing subsequence and that for any arrangement of $k > (n-1)^2$ positive integers there must be n consecutive numbers among them forming an increasing subsequence or a decreasing subsequence.

134. (a) *First solution.* Let us consider the greatest odd divisors of the chosen 101 numbers which are equal to the quotients resulting from the division of each of the numbers by the highest power of two contained in its factorization. Since there are only 100 different odd numbers not exceeding 200, among these greatest odd divisors of the 101 numbers there must be two which coincide. This means that among the 101 numbers there are two which differ from each other only in the exponents of the powers of the factor 2 contained in them. It is obvious that the greatest of these two numbers is divisible by the other.

Second solution. The assertion of the problem can also be proved with the aid of the *method of mathematical induction.* Let us show that if we choose three numbers from the four numbers 1, 2, 3 and 4 then among these three numbers there are two such that one of them is divisible by the other. To this end we can simply consider all the possible cases which can occur here. (By the way, we can even start with two numbers 1 and 2: if we "choose" two numbers from them then one of the numbers is divisible by the other.) Next we shall prove that if it is impossible to choose $n+1$ numbers from the $2n$ numbers from 1 to $2n$ so that none of the chosen numbers is divisible by any other of them then it is impossible to choose $n+2$ numbers from the first $2(n+1)$ positive integers so that none of the chosen numbers is divisible by any other of them.

Indeed, let us consider some $n+2$ numbers chosen from the first $2(n+1)$ positive integers. If this set of $n+2$ numbers does not contain the numbers $2n+1$ and $2n+2$ or if it contains only one of these numbers then there are $n+1$ numbers among them not exceeding $2n$, and, according to the hypothesis, one of these numbers must necessarily be divisible by some other of them. If the set of these $n+2$ numbers contains both numbers $2n+1$ and $2n+2$ and also contains the number $n+1$ then the numbers $n+1$ and $2n+2$ form a pair of numbers one of which is divisible by the other. Finally, if the set of these $n+2$ numbers contains the numbers $2n+1$ and $2n+2$ but does not contain the number

$n+1$, we exclude the numbers $2n+1$ and $2n+2$ and add the number $n+1$ to obtain $n+1$ numbers not exceeding $2n$ among which, according to the hypothesis, there is one number divisible by some other. If that number differs from $n+1$ we obtain a pair of numbers belonging to the $n+2$ numbers we have chosen such that one of them is divisible by the other. If that number is equal to $n+1$ then $2n+2$ is also divisible by one of the chosen numbers.

(b) To choose the required numbers we can take the following numbers: the odd numbers from 101 to 199 (50 numbers), the products of all odd numbers from 51 to 99 by 2 (25 numbers), the products of all odd numbers from 27 to 49 by 4 (12 numbers), the products of all odd numbers from 13 to 25 by 8 (7 numbers), the products of all odd numbers from 7 to 11 by 16 (3 numbers) and three more numbers $3 \cdot 32$, $5 \cdot 32$ and $1 \cdot 64$.

(c) Let us suppose that we have chosen 100 whole numbers not exceeding 200 none of which is divisible by any other. Let us prove that none of the numbers from 1 to 15 is contained among these 100 numbers.

As in the first solution of Problem 134 (a), let us consider all the greatest odd divisors of the chosen numbers. It is obvious that these divisors form the set of all odd numbers not exceeding 200 (see the solution of Problem 134 (a)). In particular, these odd divisors include the numbers 1, 3, 9, 27 and 81. Since among the numbers corresponding to these odd divisors there are not two numbers which are divisible by each other, the number containing the odd factor 27 must be divisible by a power of 2 whose exponent is not less than 1, the number containing the odd factor 9 must be divisible by a power of 2 whose exponent is not less than 2, the number containing the factor 3 must be divisible by a power of 2 with exponent not less than 3 and the number containing the factor 1 must be divisible by a power of 2 with exponent not less than 4. This means that the numbers 1, $2 = 1 \cdot 2$, 3, $4 = 1 \cdot 2^2$, $6 = 3 \cdot 2$, $8 = 1 \cdot 2^3$, 9 and $12 = 3 \cdot 2^2$ are not contained among the 200 given numbers.

In just the same way we can consider those of the given numbers whose greatest odd divisors are 5, 15 and 45 and prove that the given numbers do not contain the numbers 5, $10 = 5 \cdot 2$ and 15; similarly, the investigation of those of the given numbers whose greatest odd divisors are 7 and 21 shows that there is no number equal to 7 among the 200 numbers. Further, the investigation of those of the given numbers whose greatest odd divisors are 11 and 33 shows that the given numbers do not contain the number 11, and the investigation of those of the given numbers whose greatest odd divisors are 13 and 39 shows that the given numbers do not contain the number 13.

Remark. By analogy with the solutions of Problems 134 (a), (b) and (c), we can show that it is impossible to choose $n+1$ numbers from $2n$ (or less) first positive integers so that among them there are not two numbers divisible by each other and that, at the same time, it is possible to choose n (or less) such numbers. Besides, if $3^k < 2n < 3^{k+1}$ then among the $2n$ first positive integers there are not n numbers such that *at least one of them is less than* 2^k so that among these n numbers there are not two numbers divisible by each other, and, at the same time, it is possible to choose n such numbers the smallest of which is equal to 2^k (for instance, among the 200 first positive integers it is possible to choose 100 numbers the smallest of which is equal to 16 so that none of these numbers is divisible by another).

135. (a) Let us consider the remainders with the smallest absolute values which are obtained when the given numbers are divided by 100 (here it is meant that if the division of a number a by 100 leaves a positive remainder exceeding 50 then we consider the corresponding negative remainder $-r$, that is we represent the number a in the form $a = 100q - r$ where $0 < r < 50$). Since there are exactly 51 nonnegative integers not exceeding 50 (namely, 0, 1, 2, . . . , 50) while the number of the remainders we are considering is equal to 52, there are two among these remainders whose absolute values coincide. In case these two remainders are of one sign the *difference* of the corresponding numbers is divisible by 100; in case the remainders have opposite signs the *sum* of these numbers is divisible by 100.

(b) Let $a_1, a_2, a_3, \ldots, a_{100}$ be the given numbers (they can be arranged in an arbitrary order). Let us consider the sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \ \ldots$$
$$\ldots, \ s_{100} = a_1 + a_2 + a_3 + \ldots + a_{100}$$

Since the number of these sums is equal to 100, it follows that if none of them is divisible by 100 then there are at least two among these sums whose division by 100 leaves equal remainders (because there can only be 99 nonzero different remainders resulting from the division by 100). Let us take these two sums whose division by 100 leaves equal remainders and subtract the smallest of them from the other; this results in a sum of the form $a_{k+1} + a_{k+2} + \ldots + a_m$ which is divisible by 100.

Remark. It is clear that the method of the solution of this problem can be in just the same way used to prove that *among any n integers* (where n is an arbitrary natural number) *there are always several numbers whose sum is divisible by n.*

(c) In the case when all the numbers in question are equal (and, consequently, they are all equal to 2) the assertion stated in the problem is quite evident because the sum of any 50 of these numbers is equal to 100. If, for instance $a_1 \neq a_2$ then let us consider the following sums (cf. the solution of Problem 135 (b)):

$$s_1 = a_1, \quad s_2 = a_2, \quad s_3 = a_1 + a_2,$$
$$s_4 = a_1 + a_2 + a_3, \ \ldots, \ s_{100} = a_1 + a_2 + \ldots + a_{99}.$$

As before, we conclude *that one of these sums is divisible by* 100 *or there are two sums among them such that their division by* 100 *leaves equal remainders*; in the latter case *the difference* $s_i - s_j$ *of these two sums determines a subset of the given set of numbers the sum of whose members is divisible by* 100. (It should be noted that since $a_1 \neq a_2$ and $a_1, a_2 \leqslant 100$, the remainders resulting from the division of the "sums" $s_1 = a_1$ and $s_2 = a_2$ by 100 cannot coincide.) Further, if a sum of some of the given numbers is divisible by 100 (this sum does not include all the numbers because at least one number, for instance, the number a_{100} is not contained in it) then, since this sum is positive and is less than 200, it must be equal to 100.

(d) First of all we note that *among any four integral numbers there are always two numbers whose sum is divisible by* 2 *and that among any ten integral numbers there are always five numbers whose sum is divisible by* 5. These auxiliary assertions are similar to the one we have to prove but are of course simpler (besides, they will be used in the proof of the assertion stated in the problem). By the way, all the indicated estimates (including the result of the present problem) can be made more precise: it can be shown that among any *three* numbers there are two numbers whose sum is divisible by 2 and that among any *nine* numbers there are five numbers whose sum is divisible by 5; further, *among any* 199 *numbers there are* 100 *numbers whose sum is divisible by* 100 (see the remark at the end of the solution of the problem).

Thus, we shall prove in succession the following three facts.

1°. *Given* 3 *numbers* a_1, a_2 *and* a_3, *there are* 2 *numbers among them whose sum is divisible by* 2. This proposition is quite obvious because as these two numbers we can take the numbers which are simultaneously even or odd; it is clear that there are always two such numbers among the given three numbers.

It is clear that this simple argument is in fact based on the possibility of replacing the numbers a_1, a_2 and a_3 by the corresponding *remainders* r_1, r_2 and r_3 *resulting from the division of the given numbers by* 2: if the sum of two of these remainders is divisible by 2 then the sum of the corresponding numbers themselves is also divisible by 2. As to the numbers r_1, r_2 and r_3, they can only assume the values 0 and 1, and therefore it is clear that among them there are always two values whose sum is divisible by 2.

2°. Now let us consider 9 *numbers* $a_1, a_2, \ldots, a_9$; we assert that *there are* 5 *numbers among them whose sum is divisible by* 5. To prove the assertion we again replace the numbers a_i themselves (where $i = 1, 2, \ldots, 9$) by the corresponding remainders r_i resulting from the division of these numbers by 5. Each of the numbers r_i can only assume one of the five values 0, 1, 2, 3 and 4 (it is more convenient to consider these smaller numbers than the original numbers a_i). Besides, *if we add to all the* 9 *numbers* r_i *one and the same arbitrary number* c *or subtract* c *from the* 9 *numbers* r_i, *the remainder resulting from the division of the sum of any* 5 *of the numbers by* 5 *does not change*, and in our further argument we shall use the indicated property. (Let us agree that after the operation of "shifting" the remainders by $+c$ or by $-c$ we again replace the resultant numbers $r'_i = r_i \pm c$ by the remainders obtained when the numbers r'_i are divided by 5 and then change the notation and, as before, designate the new remainders by the letters r_i.) What has been said allows us to assume that among the 9 numbers r_i (each of which can assume one of the values 0, 1, 2, 3 and 4) the number 0 occurs more seldom than the other values because of a number $k > 0$ occurred more often among the numbers r_i we could simply subtract k from all these numbers. It is also clear that the number t of zeros must lie within the limits from 2 to 9: $2 \leqslant t \leqslant 9$ (we have $t > 1$ because among the nine remainders at least one occurs twice). It should also be noted that in the case when $t \geqslant 5$ the assertion is evident and no proof is needed because for $t \geqslant 5$ we have 5 numbers divisible by 5 whose sum is of course divisible by 5.

The further course of the argument is simple but rather lengthy. If $t = 2$ then none of the nine remainders r_i is repeated more than twice; in other words, there are 4 remainders among the nine numbers r_i each of which is repeated twice and only one remainder which occurs only once. Therefore using the same method of "shifting" the remainders by one and the same number $\pm c$ we can make this "single" remainder be equal to the number 4. Then the new remainders can be denoted and arranged as $r_1, r_2, r_3, \ldots, r_9$, so that their values are equal to the numbers 0, 0, 1, 1, 2, 2, 3, 3, 4 respectively, and, for instance, in this case the sum $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7$ is divisible by 5 (it is simply equal to 5). If $t = 3$ or $t = 4$ then let us assume that $r_1 = r_2 = r_3 = 0$ and that all the numbers $r_5, r_6, \ldots, r_9$ are different from zero; here the number r_4 can be equal to 0 or can be greater than 0 (this is of no importance). Further, in the collection of 5 numbers r_5, r_6, r_7, r_8 and r_9 (which are different from 0) there are always several (more than one!) numbers such that their sum is divisible by 5 (this can be proved by analogy with the solution of Problem 135 (b); cf. the remark at the end of the solution). Finally, to the collection of the remainders obtained in this way (their sum is divisible by 5) we can add the necessary number of zeros belonging to the set of the numbers r_1, r_2 and r_3 (it is however possible that no additional numbers are needed) to obtain a collection of five numbers whose sum is divisible by 5.

3°. Now we can show that *any collection of arbitrary* 199 *integral numbers* $a_1, a_2, \ldots, a_{199}$ *always contains* 100 *numbers whose sum is divisible by* 100. It is evident that among these 199 numbers there are always 99 pairs of numbers *simultaneously even or odd*; these pairs can be chosen from the 199 numbers in succession; after 98 pairs have been chosen there remain 3 numbers among which, by virtue of item 1° of the solution of the present problem, there are 2 numbers which can be chosen in the required manner. Let us index the 198 numbers we have chosen so that a_1 and a_2, a_3 and a_4, ..., a_{197} and a_{198} are pairs of simultaneously even or odd numbers. Further, let us replace every pair of numbers a_{2i-1} and a_{2i} (where $i = 1, 2, \ldots, 99$) by half their sum $b_i = (a_{2i-1} + a_{2i})/2$. It is evident that if from the 99 (integral) numbers b_i it is possible to choose 50 numbers whose sum is divisible by 50 then the sum of the numbers a_i corresponding to these 50 numbers b_i which we have chosen (the latter sum is twice as great as the former) is divisible by 100. Hence, for our aims it is sufficient to show that *any collection of* 99 *integral numbers contains* 50 *numbers whose sum is divisible by* 50. To this end, by analogy with the above, let us choose 49 pairs of numbers from the 99 numbers b_i so that the numbers forming each of the pair are *simultaneously even or odd*. Let us index these pairs as b_1 and b_2, b_3 and b_4, ..., b_{97} and b_{98} and then replace every pair of numbers b_{2k-1} and b_{2k} (where $k = 1, 2, \ldots, 49$) by one number c_k equal to half the sum of b_{2k-1} and b_{2k}: $c_k = (b_{2k-1} + b_{2k})/2$. Then *if a sum of some* 25 *numbers* c_k *is divisible by* 25 *then the sum of the* 50 *numbers* b_i *corresponding to them is divisible by* 50. Further application of this method is impossible because 25 is an odd number; therefore in our further argument we shall choose not pairs but *five-tuples* of numbers from the set of the numbers c_k.

As is already known, *any* 9 *numbers contain* 5 *numbers whose sum is divisible by* 5. Therefore among the 49 numbers c_k there are 5 numbers (we shall denote them as c_1, c_2, c_3, c_4 and c_5) whose sum is divisible by 5. From the remaining $49 - 5 = 44$ numbers we can choose 5 more numbers whose sum is divisible by 5 and then continue this process. On choosing in this way 8 five-tuples of numbers such that the sum of the numbers forming each five-tuple is divisible by 5, we arrive at the remaining collection of $49 - 8 \cdot 5 = 9$ numbers from which, according to item 2° of the present solution, it is possible to choose the last (the ninth) of the required five-tuples. Next we replace each of the five-tuples consisting of the numbers c_{5l-4}, c_{5l-3}, c_{5l-2}, c_{5l-1} and c_{5l} by their arithmetic mean $d_l = (c_{5l-4} + c_{5l-3} + \ldots + c_{5l})/5$. It is clear that *if we find* 5 *numbers* d_l *whose sum is divisible by* 5 *then the sum of the numbers* c_k *corresponding to them* (the latter sum is 5 times as great as the former) *will be divisible by* 25, and it is our aim to prove the possibility of the choice of such numbers c_k. To complete the proof we note that, according to item 2° of the present solution, *among the* 9 *numbers* d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8 *and* d_9 *there must be five numbers whose sum is divisible by* 5.

Remark. Items 1°, 2°, 3° of the solution and the statement of Problem 135 (d) admit of the following generalization. It can be proved that *among any* $2n - 1$ *integers* (where n is an arbitrary natural number) *there are* n *numbers whose sum is divisible by* n. The proof of this general fact is, however, much more complicated than the derivation of its special cases corresponding to $n = 2$, $n = 5$ and $n = 100$.

136. Since the sum of the numbers in question is equal to 1 and the greatest of them is equal to $1/2k$, the total number of the given numbers is not less than $2k$. We shall prove the assertion stated in the problem *by contradiction*; to this end let us suppose that *in any group of k numbers belonging to the given sequence the smallest number does not exceed half the greatest number* and then show that this assumption leads to a contradiction. Under this assumption, for the group $a_1, a_2, \ldots, a_k$ of k numbers (where $a_1 = 1/2k$ is the greatest number and a_k is the smallest number) we can write the inequality $a_k \leqslant a_1/2$. Similarly, for the group $a_k, a_{k+1}, \ldots, a_{2k-1}$ we have $a_{2k-1} \leqslant a_k/2 \leqslant a_1/2^2$, for the group $a_{2k-1}, a_{2k}, \ldots, a_{3k-2}$ (this group of numbers exists only when $3k - 2 \geqslant n$) we have $a_{3k-2} \leqslant a_{2k-1}/2 \leqslant a_1/2^3$ etc. On adding together all these inequalities we conclude that

$$S = a_1 + a_k + a_{2k-1} + a_{3k-2} + \ldots \leqslant a_1 + \frac{1}{2} a_1 + \frac{1}{2^2} a_1 + \ldots <$$
$$< \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) a_1 = 2a_1 \quad (*)$$

(the first two sums on the left involve only finite numbers of terms while the third sum $1 + 1/2 + 1/2^2 + \ldots$ is understood as an *infinite geometric progression*). Since the given sequence $a_1, a_2, a_3, \ldots$ is *nonincreasing* we have

$$\left.\begin{array}{r} a_2 + a_{k+1} + a_{2k} + a_{3k-1} + \ldots \leqslant S < 2a_1 \\ a_3 + a_{k+2} + a_{2k+1} + a_{3k} + \ldots \leqslant S < 2a_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k-1} + a_{2k-2} + a_{3k-3} + \ldots \leqslant S < 2a_k \end{array}\right\} \quad (**)$$

Finally, on adding together inequality (*) and all inequalities (**) we obtain

$$a_1 + a_2 + a_3 + \ldots + a_n < \underbrace{2a_1 + 2a_1 + \ldots + 2a_1}_{k \text{ summands}} = 2ka_1 =$$
$$= 2k \cdot \frac{1}{2k} = 1$$

whereas, by the condition of the problem, there must be $a_1 + a_2 + \ldots + a_n = 1$. We have thus arrived at a contradiction, which proves the required assertion.

137. Let us move along the given circle of crosses and noughts beginning with a nought until we come to that very nought again (for definiteness, let the motion along the circle be in clockwise direction). Then the number of all the *passages* from a nought to a nought or to a cross is equal to the total number of the noughts, that is to q. Among these passages there are a number of passages from a nought to the next nought: the number of these passages coincides with the number of pairs of noughts standing side by side, that is it coincides with the number b. It follows that the number of the passages from a nought to a cross is equal to $q - b$.

Similarly, when moving along the circle we can find the number of the passages from a cross to a nought: this number is equal to $p - a$.

Further, the number of the passages from a nought to a cross is equal to that of the passages from a cross to a nought because we stop moving along the given circle of crosses and noughts when we arrive at the initial nought. Therefore $q - b = p - a$ whence

$$a - b = p - q$$

138. It is obvious that if we have $i_k = k$ for one of the given numbers then the product under consideration is equal to zero (that is this product is *even*). Further, if we take the sequence $i_1 = 2,\ \ i_2 = 1,\ \ i_3 = 4,\ \ i_4 = 3,\ \ \ldots,\ \ i_{2m-1} = 2m,\ \ i_{2m} = 2m - 1$ (which is a permutation of the numbers $1, 2, 3, 4, \ldots, 2m - 1, 2m$) then the product

$$(1 - i_1)(2 - i_2) \ldots (2m - i_{2m}) =$$
$$= (-1) \cdot 1 \cdot (-1) \cdot 1 \cdot \ \ldots \ \cdot (-1) \cdot 1 = (-1)^m \cdot 1^m = (-1)^m$$

is *odd*. Therefore it only remains to prove that for any *odd* number $n = 2m + 1$ the product in question is always *even*.

First proof. The total number of even numbers in the sequence $1, 2, \ldots, n = 2m + 1$ *is equal to* m $(m < n/2 = (2m + 1)/2)$ (these even numbers are $2, 4, \ldots, 2m$). Therefore *the number of the even numbers in the sequence* $i_1, i_2, \ldots, i_n$ *with odd indices* *is not greater than m* (because the total number of the even numbers is equal to m) and *the number of the odd numbers with even indices is not greater than m either* (because there are only m even indices). Therefore the collection of all indices of the first and of the second type does not exhaust all $n = 2m + 1$ indices. Consequently, there is an index l such that either both i_l and l are even or both i_l and l are odd. Now, since the difference $i_l - l$ is *even* in both cases, the product of all factors $k - i_k$ where $k = 1, 2, \ldots, 2m + 1$ must be *even*.

Second proof. The sum of the factors we are considering is

$$(1 - i_1) + (2 - i_2) + (3 - i_3) + \ldots + (n - i_n) =$$
$$= (1 + 2 + 3 + \ldots + n) - (1 + 2 + 3 + \ldots + n) = 0$$

Therefore all these factors cannot be odd for an odd n (because a sum of an odd number of odd summands is always odd). Consequently, among these factors there is at least one *even* number and hence the product of the factors is *even*.

139. It is clear since each of the products x_1x_2, x_2x_3, $\ldots$, x_nx_1 is equal to $+1$ or -1, the sum of all these products can only be equal to zero when the number $n = 2m$ of the summands in that sum is even and when some m of the summands are equal to $+1$ while the other m summands are equal to -1. Since there are exactly m products among x_1x_2, x_2x_3, x_3x_4, $\ldots$, $x_{n-1}x_n$, x_nx_1 that are equal to -1, there are m changes of sign in the number sequence $x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_1$. It follows that the number m must be even (that is $m = 2k$) and hence n is divisible by 4 ($n = 4k$) because the first and the last members of this sequence coincide, and consequently the number of the changes of sign cannot be odd.

140. We shall divide the numbers in question into two groups: let the first group include all the numbers whose decimal representations contain *even* numbers of ones and let the other group include all the numbers whose decimal representations contain *odd* numbers of ones. Let A and B be two different 10-digit numbers belonging to one of the groups. Let us suppose that the decimal representations of A and B contain *one and the same* number n of ones (here $1 \leqslant n \leqslant 9$ because if $n = 0$ or $n = 10$, the numbers A and B cannot be different). If the ith digit in the decimal representation of A is equal to 1 while the ith digit of B is equal to 2, then some other digit of A (say the jth one) must be equal to 2 while the jth digit in the representation of B is equal to 1 because the decimal representations of both numbers contain the same number of ones. In this case the ith and the jth digits of the decimal representation of the sum $A + B$ are equal to 3, that is this representation must contain not less than two threes. Now let us suppose that the decimal representation of the number A contains n ones while that of B contains m ones ($m \neq n$); for definiteness, let $n > m$. Then since the numbers n and m are *simultaneously even or odd* (because A and B belong to one of the two groups), we must have $n - m \geqslant 2$, whence it follows that there are at least two numbers k and l such that the kth and the lth digits in the decimal representation of A are equal to 1 and the kth and the lth digits in the representation of B are equal to 2. Therefore it again follows that the kth and the lth digits in the decimal representation of the sum $A + B$ are equal to 3, that is the digit 3 occurs not less than twice in the decimal representation of the number $A + B$.

141. Let us write down the given five 100-digit numbers as a column and consider all the possible *pairs* of digits standing in each decimal place. The number of pairs of digits which can be formed of 5 digits is obviously equal to 10; therefore the total number of the pairs of digits is equal to $10 \cdot 100 = 1000$. In each of the columns of digits there must be contained *both* different digits 1 and 2; if in a column there is one digit 1 and four digits 2 (or, conversely, one digit 2 and four digits 1) then the number of the pairs of identical digits is equal to 6, and if a column of digits contains two digits 1 and three digits 2 (or, conversely, two digits 2 and three digits 1) then the number of the pairs of identical digits is equal to 4. Thus, the total number of all pairs of identical digits (in all 100 decimal places) can vary within the limits from $4 \cdot 100 = 400$ to $6 \cdot 100 = 600$.

On the other hand, if $A = \overline{a_1 a_2 \ldots a_{100}}$ and $B = \overline{b_1 b_2 \ldots b_{100}}$ (the bars designate the numbers consisting of the corresponding digits) are two of the given five numbers then, as we know, among the pairs of digits $a_1, b_1; a_2, b_2; \ldots; a_{100}, b_{100}$ there are exactly r pairs of identical digits. Since the number of pairs which can be formed of 5 digits is equal to 10, we thus find $10r$ pairs of identical digits. Hence, we arrive at the inequalities

$$400 \leqslant 10r \leqslant 600$$

whence it follows that $40 \leqslant r \leqslant 60$.

Remark. It is evident that if we are given five n-digit numbers then the number r defined in the same way as above must lie within the limits from $2n/5$ to $3n/5$: the inequalities $2n/5 \leqslant r \leqslant 3n/5$ can be proved by using the same argument. Let the reader investigate analogous estimates for the number r in the case when the number of the given integers differs from 5.

142. It is obviously sufficient to show that by means of the operations described in the condition of the problem *we can change any sign belonging to the first set without changing any other sign.* Indeed, if this is true then it is possible to change consecutively all those signs of the first set which are different from the signs of the second set occupying the same places to transform the first set into the second. To prove this auxiliary assertion we first of all note that it is possible to change simultaneously two arbitrary signs of the first set, say the ith sign σ_i and the jth sign σ_j. To this end it is sufficient to add to these two signs any 10 signs $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{10}}$ of the first set to form the two groups $\sigma_i, \sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{10}}$ and $\sigma_j, \sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{10}}$ of 11 signs each and to change consecutively the signs $\sigma_i, \sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{10}}$ and then to change the signs $\sigma_j, \sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{10}}$. Now, let σ_p be an arbitrary sign belonging to the first set; let us add 10 more signs $\sigma_{q_1}, \sigma_{q_2}, \ldots, \sigma_{q_{10}}$ of the first set to the sign σ_p to form the group $\sigma_p, \sigma_{q_1}, \ldots, \sigma_{q_{10}}$ of 11 signs. Next we change all these 11 signs and then apply the above technique to change simultaneously the signs σ_{q_1} and σ_{q_2}, then the signs σ_{q_3} and $\sigma_{q_4}, \ldots$ and, finally, the signs σ_{q_9} and $\sigma_{q_{10}}$. After these operations all the signs of the first set remain unchanged *except the single sign* σ_p which is changed to the opposite. We have thus proved the auxiliary assertion whence follows the assertion of the problem.

143. Let us suppose that the chess-player plays a_1 games of chess on Monday, a_2 games during Monday and Tuesday, a_3 games during the first three days etc., and, finally, a_{77} games during 77 days.

Now let us consider the following number sequence: a_1; a_2; a_3; ...; a_{77}; $a_1 + 20$; $a_2 + 20$; $a_3 + 20$; ...; $a_{77} + 20$. This sequence contains $2 \cdot 77 = 154$ numbers each of which does not exceed $132 + 20 = 152$ (the number a_{77} is not greater than $11 \cdot 12 = 132$ because a period of 77 days consists of exactly 11 weeks). Consequently, at least two of these 154 numbers are equal to each other (cf. what was said on page 9). However, there are not two numbers equal to each other among the numbers $a_1, a_2, a_3, \ldots, a_{77}$ because the chess-player plays not less than one game of chess every day. By the same reason, there are not two numbers equal to each other among the numbers $a_1 + 20$, $a_2 + 20$, $a_3 + 20, \ldots, a_{77} + 20$. Thus, for some k and l there must hold the equality

$$a_k = a_l + 20$$

which means that $a_k - a_l = 20$ whence if follows that during $k - l$ days from the $(l + 1)$th day to the kth day inclusive the chess-player plays exactly 20 games.

144. *First solution.* Let us consider the remainders resulting from the division by N of the numbers forming the sequence

$$1;\ 11;\ 111;\ \ldots;\ \underbrace{1111\ldots1}_{N \text{ ones}}$$

Since this sequence contains N numbers and the number of *different* nonzero remainders resulting from the division by N cannot be more than $N-1$, it follows that if none of the given numbers is divisible by N (if otherwise, the assertion of the problem would be proved), there are two among these numbers, say

$$K = \underbrace{11\ldots1}_{k \text{ ones}} \quad \text{and} \quad L = \underbrace{1111\ldots1}_{l \text{ ones}} \ (l > k)$$

whose division by N leaves *one and the same* remainder. In this case the difference

$$L - K = \underbrace{11\ldots1}_{l-k \text{ ones}}\underbrace{00\ldots0}_{k \text{ noughts}}$$

is divisible by N.

If N is relatively prime to 10 then the divisibility of the number $L-K = \underbrace{11\ldots1}_{l-k \text{ ones}} \cdot 10^k$ by N implies that the number $\underbrace{11\ldots1}_{l-k \text{ ones}}$ is also divisible by N.

Second solution. Let us write the fraction $1/N$ as a periodic decimal:

$$\frac{1}{N} = 0.\overline{b_1b_2\ldots b_k(a_1a_2\ldots a_l)} \text{ (where } \overline{a_1a_2\ldots a_l} \text{ is the period)}$$

According to the rule for changing a fraction to a periodic decimal, we have

$$\frac{1}{N} = \frac{\overline{b_1b_2\ldots b_ka_1a_2\ldots a_l} - \overline{b_1b_2\ldots b_k}}{\underbrace{99\ldots9}_{l \text{ nines}}\underbrace{00\ldots0}_{k \text{ noughts}}}$$

It follows that the number $A = \underbrace{999\ldots9}_{l \text{ nines}}\underbrace{00\ldots0}_{k \text{ noughts}}$ is divisible by N. Further, we have $A = 9A_1$ where $A_1 = \underbrace{11\ldots1}_{l \text{ ones}}\underbrace{00\ldots0}_{k \text{ noughts}}$. Now let us consider the number

$$B = \underbrace{11\ldots1}_{l \text{ digits}}\underbrace{00\ldots0}_{k \text{ digits}}\underbrace{11\ldots1}_{l \text{ digits}}\underbrace{00\ldots0}_{k \text{ digits}}\ldots\underbrace{11\ldots1}_{l \text{ digits}}\underbrace{00\ldots0}_{k \text{ digits}}$$

which is obtained when we write the number A_1 nine times repeatedly. It is obvious that B equals the product of the number A_1 by the number

$$\underbrace{\underbrace{100\ldots0}_{(l+k)\text{ digits}}\underbrace{100\ldots0}_{(l+k)\text{ digits}}\ldots\underbrace{100\ldots01}_{(l+k)\text{ digits}}}_{8 \text{ times}}$$

According to the test for divisibility by 9, the last number is divisible by 9. Consequently, the number B written with the aid of ones and noughts only is divisible by $9A_1 = A$ and hence it is divisible by N as well.

In case N is relatively prime to 10 the fraction $1/N$ is written as a (pure) periodic decimal of the form $1/N = 0.(\overline{a_1a_2 \ldots a_l})$ where $\overline{a_1a_2 \ldots a_l}$ is the period. Then the number B is written with the aid of ones only.

Remark. It is clear that if the decimal representation of the number A consists of p ones and that of B consists of pq ones then B is divisible by A. Therefore under the assumption that N and 10 are relatively prime we can even assert that there is an *infinitude* of numbers satisfying the condition of the problem; by the way, in the general case this also remains true.

145. Let a_N and $a_{N+1} = a_N + d$ be two *consecutive* terms of an arithmetic progression. Then the distance between the corresponding points A_N and A_{N+1} representing the numbers a_N and a_{N+1} on the number line is equal to the common difference d of the progression. Let $d > 0$; if d is not an integral number we shall denote by $\alpha = \{d\} = d - [d] > 0$ the *fractional part* of the number d (cf. page 37) and if d is an integer we shall put α equal to 1 (in all the cases the number α is equal to the *difference between d and the greatest of the integral numbers less than d*). Our aim is to construct a system of line segments of length 1 each on the number line which do not overlap and possess the property that *at least one of the points A_N falls inside one of the line segment belonging to this system.* In other words, we must prove that it is possible to exclude the case when all the points A_N are located in the *intervals* between these line segments. Suppose that this unfavourable case takes place. Then each of the line segments A_NA_{N+1} of length d consists of an integral number of line segments belonging to the given system (the total length of these segments is of course expressed by an integer) and of a number of intervals between the line segments (including two parts of such intervals). It is clear that *the total length of all these intervals between the line segments* (including the two parts of two intervals) *cannot be less than* α. Therefore if we manage to construct a system of line segments of length 1 which do not overlap and possess the property that for sufficiently large numbers N the number α is always greater than the sum of the lengths of the intervals (and of their parts) between the line segments of the system, then this system will satisfy the requirement indicated in the conditions of the problem.

What has been said allows us to make the following construction. Let us choose line segments of unit length on the positive number axis beginning with the segment (1, 2) so that the intervals between the neighbouring segments form, say, a *geometric progression with common ratio* $1/2d$. This means that after the segment (1, 2) we choose the segment (3, 4), then the segment $\left(4\frac{1}{2}, 5\frac{1}{2}\right)$, then the segment $\left(5\frac{3}{4}, 6\frac{3}{4}\right)$, then the segment $\left(6\frac{7}{8}, 7\frac{7}{8}\right)$ etc. (see Fig. 14). In this construction every line segment is twice as short as the preceding one. Then the sum of

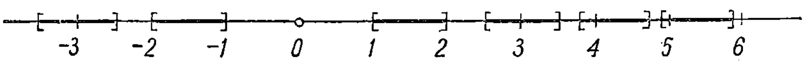


Fig. 14

the lengths of all the intervals between the segments is equal to the sum $1 + 1/2 + 1/4 + 1/8 + \ldots = 2$ and the sum R_i of the lengths of all the segments beginning with the $(i+1)$th one is equal to $1/2^i + 1/2^{i+1} + 1/2^{i+2} + \ldots = 1/2^{i-1}$. The latter sum can be made *arbitrarily small* for a sufficiently large i. Therefore for any number $\alpha = \{d\}$ (or $\alpha = 1$) corresponding to the arithmetic progression in question with positive common difference d there is always an index i_0 such that $\alpha > 1/2^{i_0-1}$, that is $\alpha > R_{i_0}$. Therefore, if N is so large that all the intervals up to the i_0th inclusive lie *to the left of* the point A_N (this can always be achieved because for $d > 0$ the sequence $A_1, A_2, A_3, \ldots, A_n, \ldots$ is not bounded and its terms *increase indefinitely*) then the line segment A_NA_{N+1} of length d can only contain the intervals beginning with the (i_0+1)th one whose lengths are $1/2^{i_0}$, $1/2^{i_0+1}$, $1/2^{i_0+2}, \ldots$. Now since the sum R_{i_0} of the lengths of all these intervals is less than α, both points A_N and A_{N+1} cannot simultaneously belong to all these intervals, that is at least one of them must necessarily fall inside a segment belonging to the system.

Up till now we have supposed that $d > 0$; to prove the assertion of the problem for the arithmetic progressions with *negative* common differences d it is sufficient to extend the system of the line segments we have constructed to the negative part of the number line. For instance, the system of line segments on the negative number axis can be taken symmetric about the origin O to the one we have constructed for the positive number axis.

146. First of all, it is evident that none of the given fractions is equal to an integral number. Indeed, if, for instance, a fraction of the form $k(m+n)/m$ (where k is equal to one of the numbers $1, 2, 3, \ldots, m-1$) were an integral number then the numbers $m+n$ and m would have common divisors (because k is less than m and cannot be divisible by m); then the number $n = (m+n) - m$ would not be relatively prime to m either.

Further, there are not two fractions among the given fractions that are equal to each other. For, if there were

$$\frac{k(m+n)}{m}=\frac{l(m+n)}{n}$$

(where k is equal to one of the numbers 1, 2, ..., $m-1$, and l is equal to one of the numbers 1, 2, ..., $n-1$) then we would have

$$\frac{k}{m}=\frac{l}{n}, \text{ that is } m=\frac{k}{l}n$$

whence it would again follow that m and n cannot be relatively prime (because l is less than n and cannot be divisible by n).

Now let us consider a positive integer A less than $m+n$. The fractions

$$\frac{m+n}{m}, \quad \frac{2(m+n)}{m}, \ldots, \quad \frac{k(m+n)}{m}$$

are less than A when $k(m+n)<Am$, that is when $k<Am/(m+n)$; the number of such fractions is obviously equal to the integral part $[Am/(m+n)]$* of the number $Am/(m+n)$. Similarly, the fractions

$$\frac{m+n}{n}, \quad \frac{2(m+n)}{n}, \ldots, \quad \frac{l(m+n)}{n}$$

are less than A when $l<An/(m+n)$; the number of such fractions is equal to the integral part $[An/(m+n)]$ of the number $An/(m+n)$. Both numbers $Am/(m+n)$ and $An/(m+n)$ are not integral because the numbers m, n, and $m+n$ are pairwise relatively prime. The sum of these two numbers is equal to A:

$$\frac{Am}{m+n}+\frac{An}{m+n}=A$$

Further, if a sum of two numbers α and β which are not integral is equal to an integral number A then

$$[\alpha]+[\beta]=A-1$$

This readily follows from what is shown in Fig. 15. Thus,

$$\left[\frac{Am}{m+n}\right]+\left[\frac{An}{m+n}\right]=A-1$$

whence we conclude that there are exactly $A-1$ fractions among the given fractions which lie within the interval $(0, A)$ on the number line.

* On the notation see page 36.

What has been proved readily implies the assertion stated in the problem. Indeed, let us first put $A = 1$; it follows that there are no fractions we are interested in among the given fractions which lie within the interval (0, 1). Further, let $A = 2$; since among the given fractions there is one fraction lying within the interval (0, 2) it follows that the interval (1, 2) also contains one of the fractions. Next we put $A = 3$; since the interval (0, 3) contains two of the fractions, that is one fraction more than the interval (0, 2) contains, it follows that there is one fraction we are interested in among the given fractions which is contained in the interval (2, 3). Continuing the same argument we complete the proof of the required proposition.

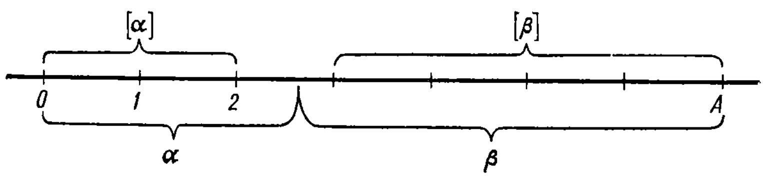


Fig. 15

147. *First solution.* If a number a_i satisfies the inequalities $1000/m \geqslant a_i > 1000/(m+1)$, the total number of positive integers not exceeding 1000 and multiple of a_i is equal to m (namely, these numbers are $a_i, 2a_i, 3a_i, \ldots, ma_i$). Therefore if we denote by k_1 the number of those of the given numbers which satisfy the inequalities $1000 \geqslant a_i > 1000/2$, by k_2 the number of those of the numbers which satisfy the inequalities $1000/2 \geqslant a_i > 1000/3$, by k_3 the number of those of the numbers which satisfy the inequalities $1000/3 \geqslant a_i > 1000/4$ etc., then the total number of positive integers not exceeding 1000 and multiple of at least one of the given numbers is equal to the sum

$$k_1 + 2k_2 + 3k_3 + \ldots$$

By the condition of the problem, all these multiples are different; consequently

$$k_1 + 2k_2 + 3k_3 + \ldots < 1000$$

Now it only remains to note that the sum of the reciprocals of all the given numbers is less than

$$k_1 \frac{1}{\frac{1000}{2}} + k_2 \frac{1}{\frac{1000}{3}} + k_3 \frac{1}{\frac{1000}{4}} + \ldots = \frac{2k_1 + 3k_2 + 4k_3 + \ldots}{1000}$$

(here we have replaced the k_1 greatest of the given numbers by $\frac{1000}{2}$, the next k_2 numbers by $\frac{1000}{3}$, the k_3 numbers following these k_2 numbers by $\frac{1000}{4}$ etc.). Since we have

$$2k_1+3k_2+4k_3+\ldots=$$
$$=(k_1+2k_2+3k_3+\ldots)+(k_1+k_2+k_3+\ldots)=$$
$$=(k_1+2k_2+3k_3+\ldots)+n<1000+n<2000$$

it follows that the sum of the reciprocals of the given numbers is less than 2.

Second solution. Let us consider another variant of the same argument. The number of the members in the sequence 1, 2, ..., 1000 divisible by an integral number a_k is obviously equal to the integral part $[1000/a_k]$ of the fraction $1000/a_k$. Since the least common multiple of any two of the numbers $a_1, a_2, \ldots, a_n$ is greater than 1000, there is no number in the sequence 1, 2, 3, ..., 1000 which is simultaneously divisible by two of the numbers $a_1, a_2, a_3, \ldots, a_n$ is equal to the sum

$$\left[\frac{1000}{a_1}\right]+\left[\frac{1000}{a_2}\right]+\left[\frac{1000}{a_3}\right]+\ldots+\left[\frac{1000}{a_n}\right]$$

Since the sequence 1, 2, 3, ..., 1000 contains 1000 numbers we must have

$$\left[\frac{1000}{a_1}\right]+\left[\frac{1000}{a_2}\right]+\left[\frac{1000}{a_3}\right]+\ldots+\left[\frac{1000}{a_n}\right]\leqslant 1000$$

Further, the integral part of a fraction differs from the fraction itself by less than 1, and therefore we have

$$\left[\frac{1000}{a_1}\right]>\frac{1000}{a_1}-1,\quad \left[\frac{1000}{a_2}\right]>\frac{1000}{a_2}-1,\ \ldots,\ \left[\frac{1000}{a_n}\right]>\frac{1000}{a_n}-1$$

Consequently

$$\left(\frac{1000}{a_1}-1\right)+\left(\frac{1000}{a_2}-1\right)+\ldots+\left(\frac{1000}{a_n}-1\right)<1000$$

that is

$$\frac{1000}{a_1}+\frac{1000}{a_2}+\frac{1000}{a_3}+\ldots+\frac{1000}{a_n}<1000+n<2000$$

and hence

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}<2$$

Remark. The estimate derived in the present problem can be made more precise. Let us consider all the multiples of the given numbers not exceeding 500. It is evident that k_1 of the given numbers are themselves greater than 500, k_2+k_3 numbers are not greater than 500 but exceed 500/2, k_4+k_5 numbers are not greater than 500/2 but exceed 500/3 etc. Using the same argument as in the first solution of the problem we conclude from what has been said that the total number of positive integers not exceeding 500 and multiple of at least one of the n given numbers is equal to

$$(k_2 + k_3) + 2(k_4 + k_5) + 3(k_6 + k_7) + \dots$$

and consequently

$$(k_2 + k_3) + 2(k_4 + k_5) + 3(k_6 + k_7) + \dots < 500$$

Now we note that the difference $500 - [(k_2 + k_3) + 2(k_4 + k_5) + 3(k_6 + k_7) + \dots]$ is equal to the number of integers which do not exceed 500 and are not multiple of any of the given numbers and that the difference $1000 - (k_1 + 2k_2 + 3k_3 + \dots)$ is equal to the number of integers which do not exceed 1000 and are not multiple of any of the given numbers. Consequently,

$$500 - [(k_2 + k_3) + 2(k_4 + k_5) + 3(k_6 + k_7) + \dots] < 1000 - (k_1 + 2k_2 + 3k_3 + \dots)$$

whence we obtain

$$(k_1 + k_2) + 2(k_3 + k_4) + 3(k_5 + k_6) + \dots < 500$$

Now it only remains to observe that

$$2k_1 + 3k_2 + 4k_3 + 5k_4 + 6k_5 + 7k_6 + \dots < (k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6 + \dots) + [(k_1 + k_2) + 2(k_3 + k_4) + 3(k_5 + k_6) + \dots] < 1000 + 500 = 1500$$

and, consequently, the sum of the reciprocals of the given numbers which is less than $(2k_1 + 3k_2 + 4k_3 + \dots)/1000$ must be less than $1\frac{1}{2}$.

Analogously, the consideration of the multiples of the given numbers not exceeding 333 shows that the sum of the reciprocals of the given numbers is even less than $1\frac{1}{5}$.

It should also be mentioned that the number 1000 in the condition of the problem can obviously be replaced by any other number.

148. Let us consider the process of changing the common fraction q/p to a repeating decimal. We have

$$\frac{q}{p} = \overline{A.a_1a_2 \dots a_ka_1a_2 \dots a_ka_1a_2 \dots}$$

where A is a whole number and $a_1, a_2, \dots, a_k$ are the digits forming the period of the decimal. It is obvious that A is equal to the quotient resulting from the division of q by p, that is we have

$$q = Ap + q_1$$

where q_1 is a remainder less than p. Further, $\overline{Aa_1}$ is the quotient resulting from the division of $10q$ by p (the number $\overline{Aa_1}$ consists of the digits of the number A and the digit a_1):

$$10q = \overline{Aa_1} \cdot p + q_2 \quad \text{where} \quad q_2 < p$$

Similarly,

$$10^2 \cdot q = \overline{Aa_1a_2} \cdot p + q_3, \ \dots, \ 10^k \cdot q = \overline{Aa_1a_2 \dots a_k} \cdot p + q_{k+1} \ \dots$$

The period of the fraction starts to appear again when the remainder q_{k+1} resulting from the division of a number of the form $10^k q$ by p coincides with the remainder q_1 resulting from the division of the number q by p: $q_{k+1} = q_1$. Hence, the number k of the digits in the period of the decimal is equal to the smallest power of 10 such that the division of $10^k q$ by p leaves the same remainder as the division of q by p. This means that the difference $10^k q - q = (10^k - 1)q$ is divisible by p, that is the difference $10^k - 1$ is divisible by p (because the number q is relatively prime to p).

Now let us suppose that k is an even number; $k = 2l$. The divisibility of the difference $10^{2l} - 1 = (10^l - 1)(10^l + 1)$ by p implies that either $10^l - 1$ or $10^l + 1$ is divisible by p. The difference $10^l - 1$ cannot be divisible by p because, if otherwise, the remainder resulting from the division of 10^l by p would coincide with the remainder resulting from the division of q by p and the period of the fraction q/p would consist of l digits and not of $k = 2l$ digits. We thus conclude that $10^l + 1$ is divisible by p.

The property we have proved shows that the sum $10^l q/p + q/p$ is an integral number. Further, we have

$$\frac{10^l q}{p} + \frac{q}{p} = \overline{Aa_1a_2 \ldots a_l . a_{l+1}a_{l+2} \ldots a_{2l}a_1a_2 \ldots a_l \ldots} + \\ + \overline{A . a_1a_2 \ldots a_la_{l+1} \ldots a_{2l} \ldots}$$

and consequently the sum of the fractions

$$\overline{0 . a_{l+1}a_{l+2} \ldots a_{2l}a_1a_2 \ldots a_l \ldots} + \overline{0 . a_1a_2 \ldots a_la_{l+1} \ldots a_{2l} \ldots}$$

is an integral number. Since each of these fractions is less than 1 and greater than 0, their sum must be equal to $1 = 0.999 \ldots$, which is only possible when

$$a_1 + a_{l+1} = 9, \quad a_2 + a_{l+2} = 9, \ \ldots, \ a_l + a_{2l} = 9$$

The last relations readily imply that

$$\frac{a_1 + a_2 + \ldots + a_{2l}}{2l} = \frac{9}{2}$$

In case k is an odd number the equality $\frac{a_1 + a_2 + \ldots + a_k}{k} = \frac{9}{2}$ is obviously impossible because the denominator of the fraction on the left-hand side of this equality is not divisible by 2.

149. The numbers of digits in the periods of the fractions a_n/p^n and a_{n+1}/p^{n+1} are equal to the smallest positive integers k and l such that $10^k - 1$ is divisible by p^n and $10^l - 1$ is divisible by p^{n+1} (see the solution of the foregoing problem). Now let us consider the difference

$$(10^l - 1) - (10^k - 1) = 10^k (10^{l-k} - 1)$$

The divisibility of this difference by p^n implies that $10^{l-k} - 1$ is divisible by p^n. Now let us show that from the divisibility of $10^{l-k} - 1$ and of $10^k - 1$ by p^n it follows that $10^d - 1$ where d is the greatest common divisor of the numbers $l - k$ and k is also divisible by p^n.

Indeed, let $l - k = qk + r$. Then we have

$$10^{l-k} - 1 = 10^{qk+r} - 1 = 10^r (10^{qk} - 1) + (10^r - 1)$$

Since the number $10^{qk} - 1 = (10^k)^q - 1^q$ is divisible by $10^k - 1$, it follows that this number is divisible by p^n, and consequently $10^r - 1$ is also divisible by p^n. In just the same way we can show that the number $10^{r_1} - 1$ where r_1 is the remainder resulting from the division of k by r is also divisible by p^n, the number $10^{r_2} - 1$ where r_2 is the remainder resulting from the division of r by r_1 is divisible by p^n, the number $10^{r_3} - 1$ where r_3 is the remainder resulting from the division of r_1 by r_2 is divisible by p^n etc.* Further, it can easily be shown that the sequence of the numbers $r, r_1, r_2, r_3, \ldots$ must end with the number d. Indeed, since $l - k$ and k are divisible by d, the number $r = (l - k) - qk$ is also divisible by d; since k and r are divisible by d the number r_1 is also divisible by d; since r and r_1 are divisible by d the number r_2 is also divisible by d etc.; consequently, all the numbers in this sequence are divisible by d. On the other hand, if r_k is the last number of that sequence (this means that r_{k-1} is exactly divisible by r_k, that is the remainder following r_k is equal to zero) then r_{k-1} is divisible by r_k; the number r_{k-2} is divisible by r_k (because both r_{k-1} and r_k are divisible by r_k); the number r_{k-3} is divisible by r_k (because both r_{k-2} and r_{k-1} are divisible by r_k) etc., and, finally, k and $l - k$ are divisible by r_k. Therefore the inequality $r_k > d$ contradicts the fact that d is the *greatest* common divisor of $l - k$ and k.

According to the definition, k is the *smallest* positive integer such that $10^k - 1$ is divisible by p^n. Therefore the divisibility of $10^d - 1$ by p^n implies that $d = k$ and $l - k$ is multiple of k, whence it follows that l is multiple of k: $l = km$.

Now let us factor the expression $10^l - 1$:

$$10^l - 1 = 10^{km} - 1 =$$
$$= (10^k - 1)(10^{(m-1)k} + 10^{(m-2)k} + \ldots + 10^k + 1)$$

* This procedure in which a sequence of consecutive remainders $r_1, r_2, r_3, \ldots$ is obtained is known as the *Euclidean algorithm*.

Since $10^k - 1$ is divisible by p^n, the division of 10^k by p^n leaves a remainder of 1; it follows that the remainder resulting from the division of $10^{2k} = 10^k \cdot 10^k$ by p^n is equal to 1, the remainder resulting from the division of $10^{3k} = 10^{2k} \cdot 10^k$ by p^n is equal to 1, and so on. Consequently, the division of each term of the sum in the parentheses by p^n leaves a remainder equal to 1, and thus when the whole sum is divided by p^n we obtain m in the remainder. It follows that if $10^k - 1$ is not divisible by p^{n+1}, then the least number l such that $10^l - 1$ is divisible by p^{n+1} is equal to pk and $10^{pk} - 1$ is divisible by p^{n+1} and is not divisible by p^{n+2} (because the expression in the parentheses is not divisible by p^2), whence we conclude that the assertion of the problem is true.

150. **(a)** Let a and b be the first and the last digits of the sought-for number N respectively. Then this number is equal to $1000a + 100a + 10b + b = 1100a + 11b = 11(100a + b)$. Since the number N is a perfect square its divisibility by 11 implies that it must be divisible by 121 as well, that is $N/11 = 100a + b$ is divisible by 11. Further, we have

$$100a + b = 99a + (a + b) = 11 \cdot 9a + (a + b)$$

and consequently $a + b$ is divisible by 11. Since neither a nor b exceeds 9 and a is not equal to 0 we must have $1 \leqslant a + b \leqslant 18$ and therefore $a + b = 11$.

It follows that

$$100a + b = 11 \cdot 9a + 11 = 11(9a + 1)$$

whence

$$\frac{N}{121} = \frac{100a + b}{11} = 9a + 1$$

Since N is a perfect square the number $N/121$ is also a perfect square. Among the numbers of the form $9a + 1$ where a varies from 1 to 9 only the number $9 \cdot 7 + 1 = 64$ is a perfect square. Consequently, $N = 121 \cdot 64 = 7744 = 88^2$.

(b) Let a be the digit in the tens place of the number in question and let b be the digit in the ones place of the number. Then this number is equal to $10a + b$, and the number written with the aid of the same digits taken in the reverse order is equal to $10b + a$. By the condition of the problem, we have $10a + b + 10b + a = 11(a + b) = k^2$.

It follows that k^2 is divisible by 11 and therefore $a + b$ is also divisible by 11. Since $a + b \leqslant 18$, this is only possible when $a + b = 11$ and $k^2 = 121$. Thus, the sought-for numbers are

$$29;\ 38;\ 47;\ 56;\ 65;\ 74;\ 83;\ 92$$

151. Let us denote by a the two-digit number formed of the first two digits of the sought-for number N and by b the number formed of the last two digits of N. Then $N = 100a + b$, and the condition of the problem yields

$$100a + b = (a + b)^2$$

whence

$$99a = (a + b)^2 - (a + b) = (a + b)(a + b - 1) \qquad (*)$$

Thus, the product $(a + b)(a + b - 1)$ must be divisible by 99. Now let us consider separately the following 5 cases which can take place here.

1°. $a + b = 99k$ and $a + b - 1 = a/k$. Since a and b are two-digit numbers, there must be $k \leqslant 2$, and it is readily seen that the relation $k = 2$ is impossible because it leads to the values $a = 99$ and $b = 99$ which do not satisfy basic equality (*). Hence we should assume that

$$k = 1, \quad a + b = 99, \quad a = a + b - 1 = 98, \quad N = 9801 = (98 + 1)^2$$

2°. $a + b = 11m$, $a + b - 1 = 9n$ and $mn = a$. In this case we have $9n = 11m - 1$. The divisibility of $11m - 1$ by 9 implies that the remainder resulting from the division of the number m by 9 is equal to 5 (it can be verified directly that if the division of m by 9 left some other remainder then $11m - 1$ would not be divisible by 9). Thus, $m = 9t + 5$, whence it follows that $9n = 99t + 54$, that is $n = 11t + 6$. Now we can write

$$a = mn = (9t + 5)(11t + 6) = 99t^2 + 109t + 30$$

Since a is a two-digit number, it readily follows that $t = 0$. Consequently, $a = 30$, $a + b = 11m = 55$, $b = 25$ and $N = 3025 = (30 + 25)^2$.

3°. $a + b = 9m$, $a + b - 1 = 11n$ and $mn = a$. The investigation analogous to that carried out for case 2° yields the single solution $N = 2025 = (20 + 25)^2$.

4°. $a + b = 33m$, $a + b - 1 = 3n$ or $a + b = 3m$, $a + b - 1 = 33n$. This case is impossible because $a + b$ and $a + b - 1$ are relatively prime numbers.

5°. $a + b - 1 = 99k$ and $a + b = a/k$. In this case we have $a + b - 1 = 99$, $a + b = 100$ and $a = (a + b)(a + b - 1)/99 = 100$, which is impossible. Thus, the only numbers satisfying the condition of the problem are 9801; 3025 and 2025.

152. (a) A four-digit number written with the aid of four even digits may begin with the digits 2, 4, 6 or 8; in other words, it lies between 1999 and 3000 or between 3999 and 5000 or between 5999 and 7000 or between 7999 and 9000. Accordingly, the square root of this number lies between 44 and 55 or between 63 and 71 or between 77 and 84 or between 89 and 95. It should also be noted that since we have $(10x + y)^2 = 100x^2 + 20xy + y^2$, the digit in the tens place of the number $(10x+y)^2$ and the digit in the tens place of the number y^2 are simultaneously even or odd in the case when $0 \leqslant y \leqslant 9$. Therefore the last digit of the square root of the sought-for number cannot be equal to 4 or to 6.

Since the square root of the sought-for number is even it can be equal only to one of the following 10 numbers:

$$48;\ 50;\ 52;\ 68;\ 70;\ 78;\ 80;\ 82;\ 90;\ 92$$

It can be verified directly that the numbers satisfying the condition of the problems are

$$68^2 = 4624;\quad 78^2 = 6084;\quad 80^2 = 6400;\quad 92^2 = 8464$$

(b) The argument analogous to that used in the solution of Problem 152 (a) shows that there are no four-digit numbers written with the aid of four odd digits which are perfect squares.

153. (a) Let us denote the digits in the hundreds, tens and ones places of the sought-for number N as x, y, and z respectively; then we have $N = 100x + 10y + z$. The condition of the problem yields the relation

$$100x + 10y + z = x! + y! + z!$$

Since $7! = 5040$ is a four-digit number, none of the digits of the number N can exceed 6. Consequently, the number N itself does not exceed 700, whence it follows that none of its digits can exceed 5 (because $6! = 720 > 700$). Further, at least one digit of the number N is equal to 5 because N is a three-digit number and $3 \cdot 4! = 72 < 100$. It is clear that x cannot be equal to 5 since we have $3 \cdot 5! = 360 < 500$. It also follows that x cannot exceed 3. Further, we can assert that x does not exceed 2 since $3! + 2 \cdot 5! = 246 < 300$. Further, the number 255 does not satisfy the condition of the problem, and if only one digit of the sought-for number is equal to 5 then x cannot exceed 1 because $2! + 5! + 4! = 146 < 200$. Moreover, since $1! + 5! + 4! = 145 < 150$ we conclude that y cannot exceed 4; consequently z is equal to 5 because at least one of the digits of the number N must be equal to 5. Thus, we have $x = 1$, $4 \geqslant y \geqslant 0$ and $z = 5$, which allows us to easily find the single solution of the problem: $n = 145$.

(b) The sought-for number N cannot consist of more than three digits because even $4 \cdot 9^2 = 324$ is a three-digit number. This allows us to write $N = 100x + 10y + z$ where x, y and z are the digits of the number N; here x can be equal to 0 and it is even possible that x and y are simultaneously equal to 0.

The condition of the problem implies $100x + 10y + z = x^2 + y^2 + z^2$ whence

$$(100 - x)x + (10 - y)y = z(z - 1) \qquad (*)$$

From the last equality it follows that $x = 0$ because, if otherwise, the number on the left-hand side of the equality would not be less than 90 (in case $x \geqslant 1$ we have $100 - x \geqslant 90$ and $(10 - y)y \geqslant 0$) whereas the number on the right-hand side would not be greater than $9 \cdot 8 = 72$ (since $z \leqslant 9$). Consequently, equation (*) has the form $(10 - y)y = z(z - 1)$. It can easily be verified that the last equality cannot be fulfilled for any positive integers z and y not exceeding 9 unless $y \neq 0$. If $y = 0$ we have a single solution: it is obvious that in this case $z = 1$.

Thus, the only number satisfying the condition of the problem is $N = 1$.

154. (a) It is evident that the sought-for number N cannot have more than four digits because the sum of the digits of a five-digit number does not exceed $5 \cdot 9 = 45$ and $45^2 = 2025$ is a four-digit number. Further, since $4 \cdot 9 = 36$ and $36^2 = 1296$, the first digit of N does not exceed 1 in case N is a four-digit number. But we have $1 + 3 \cdot 9 = 28$, and $28^2 = 784$ is a three-digit number, whence it follows that N cannot be a four-digit number. Thus we can assume that $N = 100x + 10y + z$ where x, y and z are the digits of the sought-for number; it is possible that $x=0$ or even $x=y=0$.

The condition of the problem can now be written in the form

$$100x + 10y + z = (x + y + z)^2$$

whence

$$99x + 9y = (x + y + z)^2 - (x + y + z) = (x + y + z)(x + y + z - 1)$$

We thus see that either $x + y + z$ or $x + y + z - 1$ is divisible by 9 (it is impossible that each of these two numbers is divisible by 3 since they are relatively prime). Besides, $1 \leqslant x + y + z \leqslant 27$.

Now let us consider separately the following six cases which can take place here.

1°. $x + y + z - 1 = 0$; $99x + 9y = 0$; $x = y = 0$, $z = 1$; $N = 1$.

2°. $x + y + z = 9$; $99x + 9y = 9 \cdot 8 = 72$; $x = 0$, $9y = 72$, $y = 8$, $z = 1$; $N = 81 = (8 + 1)^2$.

3°. $x + y + z - 1 = 9$; $99x + 9y = 9 \cdot 10 = 90$, $x = 0$, $9y = 90$, which is impossible.

4°. $x + y + z = 18$; $99x + 9y = 18 \cdot 17 = 306$, $x = 3$, $y = 1$, $z = 18 - (3 + 1) = 14$, which is impossible.

5°. $x + y + z - 1 = 18$; $99x + 9y = 19 \cdot 18 = 342$, $x=3$, $y=5$, $z = 19 - (3 + 5) = 11$, which is impossible.

6°. $x + y + z = 27$; $99x + 9y = 27 \cdot 26 = 702$, $x = 7$, $y = 1$, $z = 27 - (7 + 1) = 19$, which is impossible.

Thus, the condition of the problem is satisfied only by the numbers 1 and 81.

(b) A cube of a three-digit number consists of not more than nine digits; therefore the sum of the digits of the cube of a three-digit number does not exceed $9\cdot 9 = 81 < 100$. It follows that the sought-for number cannot have three digits; it can similarly be proved that it cannot contain more than three digits either. Thus, the sought-for number must contain one or two digits.

A cube of a two-digit number cannot have more than six digits; therefore the sum of the digits of the cube does not exceed $6\cdot 9 = 54$. Thus, the sought-for number cannot exceed 54. Further, if a cube of a number not exceeding 54 has six digits, its first digit must be equal to 1; therefore the sum of the digits of the cube does not exceed $5\cdot 9 + 1 = 46$. Hence, the sought-for number does not exceed 46.

If a number does not exceed 46, its cube consists of not more than five digits, and since the cube is less than 99 999, the sum of the digits of the cube does not exceed $4\cdot 9 + 8 = 44$. Since the cube of the number 44 is a five-digit number whose last digit is equal to 4, the number 44 also exceeds the sum of the digits of its cube. Thus, the sought-for number does not exceed 43.

Further, since the remainder resulting from the division by 9 of the sum of the digits of any number coincides with the remainder resulting from the division by 9 of the number itself, the division of the sought-for number and of its cube by 9 must leave the same remainders. But this is only possible when the division of the sought-for number by 9 leaves a remainder equal to -1, 0 or 1.

Thus, the sought-for number does not exceed 43 and its division by 9 leaves a remainder equal to -1, 0 or 1. These conditions are satisfied only by the following 13 numbers:

$$1;\ 8;\ 9;\ 10;\ 17;\ 18;\ 19;\ 26;\ 27;\ 28;\ 35;\ 36;\ 37$$

The direct verification shows that among them the numbers satisfying the condition of the problem are

$$1\,(1^3 = 1);\quad 8\,(8^3 = 512);\quad 17\,(17^3 = 4193);$$
$$18\,(18^3 = 5832);\quad 26\,(26^3 = 17\,576);\quad 27\,(27^3 = 19\,683)$$

155. (a) We can readily check that for $x < 5$ the given equation has the only solutions $x = 1$, $y = \pm 1$ and $x = 3$, $y = \pm 3$. Now let us prove that there are no solutions for $x \geqslant 5$. Indeed, the expression $1! + 2! + 3! + 4! = 33$ ends with the digit 3 while all the factorials $5!, 6!, 7!, \ldots$ end with noughts. Consequently, for $x \geqslant 5$ the last digit of the sum $1! + 2! + \ldots + x!$ is equal to 3 and therefore this sum cannot be equal to a square of a whole number y (because a square of a whole number cannot end with 3).

(b) Let us consider the following two cases that can take place here:

1°. z is an even number: $z = 2n$. This case can easily be reduced to the foregoing problem because $y^{2n} = (y^n)^2$. Thus, for an even z we have the following solutions:

$$x = 1; \quad y = \pm 1; \quad z \text{ is an arbitrary even number}$$

and

$$x = 3; \quad y = \pm 3; \quad z = 2$$

2°. The number z is odd. If $z = 1$ then we can take any value of x, and in this case $y = 1! + 2! + \ldots + x!$ Now let $z \geqslant 3$. We have $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! = 46\,233$. The number 46 233 is divisible by 9 and is not divisible by 27 while the number $n!$ is divisible by 27 for $n \geqslant 9$. The sum $9! + 10! + \ldots + x!$ is divisible by 27; however, since $1! + 2! + \ldots + 8!$ is divisible by 9 and is not divisible by 27 the entire sum $1! + 2! + \ldots + x!$ is divisible by 9 and is not divisible by 27 for $x \geqslant 8$. For the number y^z to be divisible by 9 it is necessary that y should be divisible by 3. In that case y^z is divisible by 27 (because $z \geqslant 3$), and consequently there are no integral solutions for $x \geqslant 8$ and $z \geqslant 3$. Now it remains to consider the case $x < 8$. We have $1! = 1 = 1^z$ where z is any natural number; further, $1! + 2! = 3$, that is this sum cannot be equal to any integral power (with exponent different from 1) of any natural number. We also have $1! + 2! + 3! = 3^2$ and

$$1! + 2! + 3! + 4! = 33$$
$$1! + 2! + \ldots + 5! = 153$$
$$1! + 2! + \ldots + 6! = 873$$
$$1! + 2! + \ldots + 7! = 5913$$

None of the numbers 33; 153; 873 and 5913 is equal to an integral power (with exponent different from 1) of any natural number. Hence, for odd z we have the following solutions only:

$$x = 1, \quad y = 1, \quad z \text{ is an arbitrary odd number}$$

and

$$x \text{ is an arbitrary natural number}, \; y = 1! + 2! + \ldots + x!, \; z = 1$$

156. Let

$$a^2 + b^2 + c^2 + d^2 = 2^n$$

We shall denote by p the greatest exponent of the power of 2 by which all the four numbers a, b, c and d are divisible. On cancelling both members of the given equality by 2^{2p} we obtain

$$a_1^2 + b_1^2 + c_1^2 + d_1^2 = 2^{n-2p}$$

where there is at least one odd number among the four numbers a_1, b_1, c_1 and d_1.

If among the four numbers a_1, b_1, c_1, and d_1 there is one or three odd numbers, then the number $a_1^2+b_1^2+c_1^2+d_1^2$ is odd and the equality $a_1^2+b_1^2+c_1^2+d_1^2=2^{n-2p}$ cannot be fulfilled. If among the numbers a_1, b_1, c_1 and d_1 there are two odd numbers, say $a_1=2k+1$ and $b_1=2l+1$, while the other two numbers $c_1=2m$ and $d_1=2n$ are even, then we have

$$a_1^2+b_1^2+c_1^2+d_1^2=4k^2+4k+1+4l^2+4l+1+4m^2+4n^2=$$
$$=2[2(k^2+k+l^2+l+m^2+n^2)+1]$$

The last relation contradicts the condition that the number $a_1^2+b_1^2+c_1^2+d_1^2=2^{n-2p}$ has no odd divisors (the expression in square brackets cannot be equal to 1 because, if otherwise, we would have $k=l=m=n=0$, $c_1=d_1=0$ and $c=d=0$). In case all the four numbers $a_1=2k+1$, $b_1=2l+1$, $c_1=2m+1$ and $d_1=2n+1$ are odd we have

$$a_1^2+b_1^2+c_1^2+d_1^2=$$
$$=4k^2+4k+1+4l^2+4l+1+4m^2+4m+1+4n^2+4n+1=$$
$$=4[k(k+1)+l(l+1)+m(m+1)+n(n+1)+1]$$

A product of two consecutive whole numbers is always even (because one of the factors must necessarily be even). Consequently, the expression in the square brackets is odd and hence it is equal to 1. Thus, $n-2p=2$, $n=2p+2$ and $k=l=m=n=0$, $a_1=b_1=c_1=d_1=1$, $a=b=c=d=2^p$.

We see that if n is an odd number then 2^n cannot be written as a sum of four squares; if the number n is even ($n=2p$) then 2^n admits of only one expansion

$$2^{2p}=(2^{p-1})^2+(2^{p-1})^2+(2^{p-1})^2+(2^{p-1})^2$$

157. (a) *First solution.* The equation

$$x^2+y^2+z^2=2xyz$$

is satisfied by the values $x=0$, $y=0$, $z=0$. Besides, if one of the numbers x, y and z is equal to 0 then the other two numbers must also be equal to 0 because in this case the sum of their squares is equal to 0.

Now let us suppose that all the three numbers x, y and z satisfying the given equation are different from 0. These numbers can be represented in the form

$$x=2^\alpha x_1, \quad y=2^\beta y_1, \quad z=2^\gamma z_1$$

where x_1, y_1 and z_1 are odd numbers (if one of the numbers x, y and z is odd then the corresponding exponent of the power of 2 is equal to 0).

Since x, y and z play equivalent roles in the given equation we can assume that x is divisible by the lowest power of 2 and z is divisible by the highest power of 2 (this assumption does not restrict the generality of the argument), that is we suppose that

$$\alpha \leqslant \beta \leqslant \gamma$$

Let us determine the exponent of the power of 2 by which the left-hand member of the equality is divisible.

1°. If $\alpha < \beta \leqslant \gamma$ or $\alpha = \beta = \gamma$ then, on taking $2^{2\alpha}$ out of the brackets, we obtain in the brackets a sum of one odd and two even numbers or a sum of three odd numbers respectively, that is this sum is equal to an odd number.

2°. In case $\alpha = \beta < \gamma$ we can write

$$x = 2^{\alpha}(2k+1), \quad y = 2^{\alpha}(2l+1), \quad z = 2^{\alpha} \cdot 2m$$

Then we have

$$x^2 + y^2 + z^2 = 2^{2\alpha}[(2k+1)^2 + (2l+1)^2 + (2m)^2] =$$
$$= 2^{2\alpha}(4k^2 + 4k + 1 + 4l^2 + 4l + 1 + 4m^2) =$$
$$= 2^{2\alpha+1}[2(k^2 + l^2 + m^2 + k + l) + 1]$$

Hence, in this case the sum obtained in the brackets (after $2^{2\alpha+1}$ has been taken out of the brackets) is an odd number.

On the other hand, the right-hand member of the equality $x^2 + y^2 + z^2 = 2xyz$ is divisible by $2^{\alpha+\beta+\gamma+1}$, and the left-hand member of the equality must be divisible by the same power of 2 as the right-hand member.

It follows that in case 1° there must be $2\alpha = \alpha + \beta + \gamma + 1$. Since $\alpha \leqslant \beta \leqslant \gamma$, the last equality implies the inconsistent relation $2\alpha \geqslant 3\alpha + 1$ which cannot hold.

It also follows that in case 2° we must have $2\alpha + 1 = \alpha + \beta + \gamma + 1$. Since $\alpha = \beta < \gamma$, this implies the inequality $2\alpha + 1 > 3\alpha + 1$ which cannot hold either.

Consequently, the equation $x^2 + y^2 + z^2 = 2xyz$ has no integral solutions other than the solution $x = 0$, $y = 0$, $z = 0$.

Second solution. Since the sum of the squares of the numbers x, y and z is even we conclude that either all the numbers are even or one of them is even while the other two are odd. However, in the latter case the sum $x^2 + y^2 + z^2$ is divisible by 2 and is not divisible by 4 whereas the product $2xyz$ is divisible by 4, which is impossible (cf. the first solution of the problem). Hence, we can assume that the numbers x, y and z are even: $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$. On substituting these values into the original equation and cancelling by 4 we obtain

$$x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1$$

In just the same way, repeating the same argument for the last equation, we conclude that all the three numbers x_1, y_1 and z_1 are even. Therefore we can put $x_1 = 2x_2$, $y_1 = 2y_2$ and $z_1 = 2z_2$ and write the equation

$$x_2^2 + y_2^2 + z_2^2 = 8x_2y_2z_2$$

for the numbers $x_2 = x_1/2 = \frac{x}{4}$, $y_2 = \frac{y}{4}$, $z_2 = \frac{z}{4}$. As before, from this equation we conclude that the numbers x_2, y_2 and z_2 are also even.

Continuing the same process we conclude that all the numbers

$$x,\ y,\ z;\quad x_1 = \frac{x}{2},\quad y_1 = \frac{y}{2},\quad z_1 = \frac{z}{2};\quad x_2 = \frac{x}{4},$$

$$y_2 = \frac{y}{4},\quad z_2 = \frac{z}{4};\quad x_3 = \frac{x}{8},\quad y_3 = \frac{y}{8},\quad z_3 = \frac{z}{8};\ \ldots$$

$$\ldots;\ x_k = \frac{x}{2^k},\quad y_k = \frac{y}{2^k},\quad z_k = \frac{z}{2^k};\ \ldots$$

are even (the numbers x_k, y_k and z_k must satisfy the equation $x_k^2 + y_k^2 + z_k^2 = 2^{k+1}x_ky_kz_k$). But this is only possible when $x = y = z = 0$.

(b) Using the same argument we can show that the only integral solution of the equation $x^2 + y^2 + z^2 + v^2 = 2xyzv$ is $x = 0$, $y = 0$, $z = 0$, $v = 0$.

Here it is necessary to consider separately the case when the highest powers of 2 by which x, y, z and v are divisible have equal exponents, that is the case when

$$x = 2^\alpha(2k+1),\qquad y = 2^\alpha(2l+1)$$

$$z = 2^\alpha(2m+1),\qquad v = 2^\alpha(2n+1)$$

where α is a nonnegative integral number and k, l, m and n are some integers.

In this case we have

$$\begin{aligned}x^2 + y^2 + z^2 + v^2 &= 2^{2\alpha}[(4k^2 + 4k + 1) + (4l^2 + 4l + 1) + \\ &\quad + (4m^2 + 4m + 1) + (4n^2 + 4n + 1)] = \\ &= 2^{2\alpha+2}(k^2 + k + l^2 + l + m^2 + m + n^2 + n + 1) = \\ &= 2^{2\alpha+2}[k(k+1) + l(l+1) + m(m+1) + n(n+1) + 1]\end{aligned}$$

The expression in the square brackets must necessarily be odd (cf. page 225). Therefore the exponent of the highest power of 2 by which the left-hand member of the equality is divisible is equal to $2\alpha+2$. As to the right-hand member, it is divisible by $2^{4\alpha+1}$. Consequently there must hold the equality $2\alpha+2=4\alpha+1$, which is impossible for integral α.

The second solution of Problem 157 (b) is analogous to the second solution of Problem 157 (a); let the reader consider this solution.

158. (a) Let x, y and z be three positive integers satisfying the equation

$$x^2+y^2+z^2=kxyz \qquad (*)$$

First of all let us show that it is allowable to assume (without loss of generality) that the inequalities

$$x\leqslant\frac{kyz}{2}, \quad y\leqslant\frac{kxz}{2}, \quad z\leqslant\frac{kxy}{2} \qquad (**)$$

take place. This assumption simply means that none of the summands on the left-hand side of equation (*) exceeds half the right-hand side. Indeed, if, for instance, we had $z>\frac{kxy}{2}$ then we could replace the numbers x, y, z by the smaller numbers x, y and $z_1=kxy-z$ which, as can easily be seen, also satisfy equation (*):

$$x^2+y^2+(kxy-z)^2=kxy(kxy-z)$$

If one of the new numbers is again greater than the product of the other two numbers multiplied by $k/2$ then we can again carry out an analogous replacement and continue this process until we arrive at a triple of numbers for which conditions (**) are fulfilled (after this the continuation of the process no longer leads to further decrease of the numbers x, y and z).

Let us suppose that $x\leqslant y\leqslant z$. It is readily seen that the inequalities $y\leqslant z\leqslant kxy/2$ imply

$$1\geqslant\frac{kx}{2}, \text{ that is } kx\geqslant 2$$

Equation (*) can obviously be rewritten in the form

$$x^2+y^2+\left(\frac{kxy}{2}-z\right)^2=\left(\frac{kxy}{2}\right)^2$$

Since $z\leqslant kxy/2$ we see that when the number z in the left-hand member of the last equality is replaced by $y\leqslant z$ the left-hand member increases (in the case when $y=z$ it does not change). Consequently,

$$x^2+y^2+\left(\frac{kxy}{2}-y\right)^2\geqslant\frac{k^2x^2y^2}{4}$$

On opening the parentheses in the last inequality we obtain

$$x^2 + 2y^2 \geqslant kxy^2$$

By the hypothesis, $x \leqslant y$ and therefore we must have

$$y^2 + 2y^2 \geqslant kxy^2$$

that is

$$kx \leqslant 3$$

Thus, $2 \leqslant kx \leqslant 3$, that is kx is equal to 2 or to 3. In the case when $kx = 2$ equation (*) takes the form

$$x^2 + y^2 + z^2 = 2yz, \quad \text{that is} \quad x^2 + (y - z)^2 = 0$$

whence we conclude that $x = 0$ and that kx is not equal to 2 but is equal to 0. Consequently, there must be $kx = 3$, whence it follows that k can only be equal to 1 or to 3. Simple examples (cf. the solution of Problem 158 (b)) show that these values of k are admissible.

(b) Let us continue the argument used in the solution of Problem 158 (a). In that solution we had the inequality $x^2 + 2y^2 \geqslant kxy^2$; since $kx = 3$, this inequality can be rewritten in the form

$$x^2 + 2y^2 \geqslant 3y^2, \quad \text{that is} \quad x^2 \geqslant y^2$$

Since we assumed that $x \leqslant y$, it follows that $x = y$. Now we put $x = y$ and $kx = 3$ in the original equation (*) to obtain

$$2x^2 + z^2 = 3xz, \quad \text{that is} \quad (z - x)(z - 2x) = 0$$

Thus, we have $z = x$ or $z = 2x$. Since $z \leqslant kxy/2 = 3y/2 = 3x/2$ the number z cannot be equal to $2x$, and consequently $z = x$.

Thus, if conditions (**) are fulfilled we must have $x = y = z$. Now, since $kx = 3$, the number x can only be equal to 1 or 3. Accordingly, we obtain the following two solutions of equation (*):

$$x = y = z = 1 \quad (k = 3)$$

and

$$x = y = z = 3 \quad (k = 1)$$

As was shown in the solution of Problem 158 (a), any triple of numbers x, y, z satisfying equation (*) can be transformed with the aid of consecutive substitutions of the form $z_1 = kxy - z$ into a triple of numbers satisfying inequalities (**). Now, since $z_1 = kxy - z$ implies $z = kxy - z_1$ we see that every solution of equation (*) can be obtained from the smallest solutions written above by means of consecutive substitutions of the form $z_1 = kxy - z$. In particular, in this way we obtain the following solutions of equation (*) not exceeding 1000:

1°. The case $k = 3$.

x	1	1	1	1	1	1	1	1	2	2	2	5	5
y	1	1	2	5	13	34	89	233	5	29	169	13	29
z	1	2	5	13	34	89	233	610	29	169	985	194	433

2°. The case $k = 1$.

x	3	3	3	3	3	3	3	6	6	15
y	3	3	6	15	39	102	267	15	87	39
z	3	6	15	39	102	267	699	87	507	582

(The fact that the solutions corresponding to the value $k = 1$ are obtained from the solutions corresponding to the value $k = 3$ by means of the multiplication of the numbers x, y and z by 3 is a direct consequence of the relations connecting the *smallest* solutions of the equations $x^2 + y^2 + z^2 = xyz$ and $x^2 + y^2 + z^2 = 3xyz$.)

159. The equality $x^3 = 2(y^3 + 2z^3)$ (where x, y and z are integers) implies that x is even: $x = 2x_1$; this allows us to rewrite the given equation in the form

$$8x_1^3 - 2y^3 - 4z^3 = 0, \quad \text{that is} \quad 4x_1^3 - y^3 - 2z^3 = 0$$

Since $y^3 = 2(2x_1^3 - z^3)$, the number y is even: $y = 2y_1$; therefore we have

$$4x_1^3 - 8y_1^3 - 2z^3 = 0, \quad \text{that is} \quad 2x_1^3 - 4y_1^3 - z^3 = 0$$

Finally, since $z^3 = 2(x_1^3 - 2y_1^3)$ the number z is also even: $z = 2z_1$, and we obtain

$$2x_1^3 - 4y_1^3 - 8z_1^3 = 0, \quad \text{that is} \quad x_1^3 - 2y_1^3 - 4z_1^3 = 0$$

Hence, if x, y, z is a solution of the original equation then all the three numbers x, y and z are even and their halves $x_1 = x/2$, $y_1 = y/2$ and $z_1 = z/2$ satisfy exactly the same equation:

$$x_1^3 - 2y_1^3 - 4z_1^3 = 0$$

It follows that x_1, y_1 and z_1 are also even numbers: $x_1 = 2x_2$, $y_1 = 2y_2$ and $z_1 = 2z_2$; besides, the numbers x_2, y_2, and z_2 also satisfy the original equation, that is they are also even etc. In this way we finally conclude that the integral numbers x, y, and z are divisible by *any* power of 2, which is obviously possible only

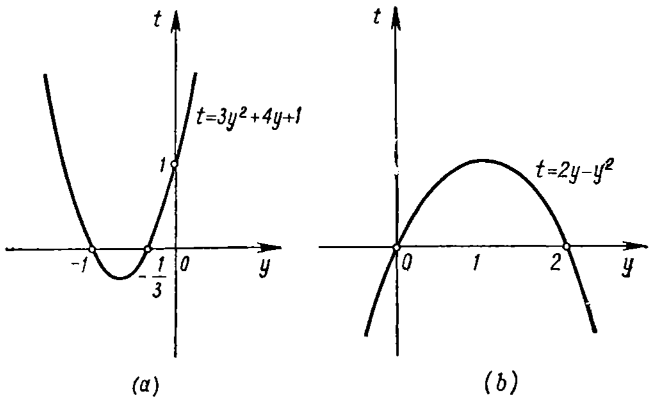


Fig. 16

in the case when they all are equal to zero. Thus, the original equation has a single solution, namely

$$x = y = z = 0$$

160. Let us multiply both members of the equation by 4 and add 1 to them; this results in the equivalent equation

$$(2x + 1)^2 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$$

whose left-hand member is a perfect square. Further, we have

$$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (4y^4 + 4y^3 + y^2) + (3y^2 + 4y + 1) =$$
$$= (2y^2 + y)^2 + (3y^2 + 4y + 1) = (P(y))^2 + Q(y)$$

Since the quadratic trinomial $Q(y) = 3y^2 + 4y + 1$ possesses (real) roots $y_1 = -1$ and $y_2 = -1/3$ it assumes positive values for all integral values of y different from $y = -1$ (see the graph of the function $t=3y^2+4y+1$ in Fig. 16*a*). Therefore $(2x+1)^2 > (P(y))^2 = (2y^2 + y)^2$.

On the other hand,

$$4y^4 + 4y^3 + 4y^2 + 4y + 1 =$$
$$= (2y^2 + y + 1)^2 + (2y - y^2) = (P_1(y))^2 + Q_1(y)$$

The graph of the function $Q_1(y) = 2y - y^2$ is shown in Fig. 16*b*; the roots of the quadratic binomial $Q_1(y)$ are equal to 0 and 2.

Therefore $Q_1(y) < 0$ for all integral values of y different from 0, 1 and 2, whence $(2x+1)^2 < (P_1(y))^2 = (2y^2+y+1)^2$.

Thus, for all integral values of y different from -1, 0, 1 and 2 there hold the inequalities

$$(2y^2+y+1)^2 > (2x+1)^2 > (2y^2+y)^2$$

This means that for such y the number $(2x+1)^2$ lies between the squares of the two consecutive whole numbers $Q(y)$ and $Q_1(y)$, and therefore $2x+1$ cannot be equal to an integral number.

Thus, in case y is an integral number, the number x can be integral only when y is equal to -1, 0, 1 or 2, that is when the right-hand side of the original equation is equal to 0, 0, 4 or 30 respectively. It now remains to solve 3 quadratic equations of the form

$$x^2+x=c \quad \text{where} \quad c \quad \text{is equal to 0, 4 or 30} \qquad (*)$$

These equations have the following integral roots:

$$x=0 \text{ and } x=-1 \text{ for } c=0; \; x=5 \text{ and } x=-6 \text{ for } c=30;$$
$$\text{for } c=4 \text{ equation } (*) \text{ has no integral roots}$$

Hence, finally, we arrive at the following set of integral solutions of the given equation:

$$(0, -1), (-1, -1); \quad (0, 0), (-1, 0); \quad (5, 2), (-6, 2)$$

(here the notation (a, b) means that $x=a$ and $y=b$; the total number of the solutions is equal to 6).

161. If $y=1$ we obtain the quadratic equation

$$x^2+(x+1)^2=(x+2)^2, \quad \text{that is} \quad x^2-2x-3=0$$

This equation has a single positive integral root $x=3$ (the other root $x=-1$ of the equation is negative). Now, let $y>1$. It should be noted that since the numbers x^{2y} and $(x+2)^{2y}$ are simultaneously even or odd, the number $(x+1)^{2y}=(x+2)^{2y}-x^{2y}$ must be even; therefore the number $x+1$ is *even*: $x+1=2x_1$. Further, we have

$$(2x_1)^{2y}=(x+1)^{2y}=(x+2)^{2y}-x^{2y}=(2x_1+1)^{2y}-(2x_1-1)^{2y} \qquad (*)$$

Using Newton's binomial formula we open the parentheses on the right-hand side to obtain

$$(2x_1)^{2y}=$$
$$=2\left[C(2y,\ 1)(2x_1)^{2y-1}+C(2y,\ 3)(2x_1)^{2y-3}+\ \dots\ +C(2y,\ 1)(2x_1)\right]$$

Since $C(2y, 1) = 2y$, this relation can be rewritten in the form

$$2 \cdot 2y \cdot 2x_1 =$$
$$= (2x_1)^{2y} - 2 \cdot 2y\,(2x_1)^{2y-1} - 2 \cdot C(2y, 3)\,(2x_1)^{2y-3} - \dots -$$
$$- 2 \cdot C(2y, 3)\,(2x_1)^3$$

(it should be noted that $2y \geqslant 4$ since $y \geqslant 2$). It is obvious that all the terms on the right-hand side are divisible by $(2x_1)^3$, whence it follows that *y must be divisible by x_1^2*.

Further, on dividing both members of equality (*) by $(2x_1)^{2y}$ we obtain

$$1 = \left(1 + \frac{1}{2x_1}\right)^{2y} - \left(1 - \frac{1}{2x_1}\right)^{2y}$$

which implies $(1 + 1/2x_1)^{2y} = 1 + (1 - 1/2x_1)^{2y} < 2$. On the other hand, by Newton's binomial formula,

$$\left(1 + \frac{1}{2x_1}\right)^{2y} = 1 + 2y \cdot \frac{1}{2x_1} + C(2y, 2)\left(\frac{1}{2x_1}\right)^2 + \dots =$$
$$= 1 + \frac{y}{x_1} + \dots > 1 + \frac{y}{x_1}$$

Consequently,

$$1 + \frac{y}{x_1} < 2, \quad \text{that is} \quad \frac{y}{x_1} < 1 \quad \text{whence} \quad y < x_1$$

which contradicts the divisibility of y by x_1^3. Therefore the given equation has no solutions such that $y > 1$, and all the solutions are those found above: $y = 1$, $x = 3$.

162. Let us denote $\underbrace{\sqrt{x + \sqrt{x + \dots + \sqrt{x}}}}_{y \text{ square roots}}$ as $A_y(x)$. Then we can write

$$x + A_{y-1}(x) = x + \underbrace{\sqrt{x + \dots + \sqrt{x}}}_{y-1 \text{ square roots}} = z^2 \quad \text{that is} \quad A_{y-1}(x) = z^2 - x$$

Hence, if the number $A_y(x) = z$ is integral then the number $A_{y-1}(x) = z^2 - x$ is also integral; in this case the numbers $A_{y-2}(x) = (z^2 - x)^2 - x$, $A_{y-3}(x) \dots$, $A_1(x) = \sqrt{x}$ are also integral. Since $\sqrt{x} = t$ is an integral number it follows that $x = t^2$ (where t is an integral number).

It is clear that for any integral value of t the numbers $x = t^2$, $y = 1$, $z = t$ are solutions of the given equation. Now let $y > 1$. In this case the numbers

$$A_1(x) = \sqrt{x} = t \quad \text{and} \quad A_2(x) = \sqrt{x + \sqrt{x}} = \sqrt{t^2 + t} = \sqrt{t(t+1)}$$

must be integral; since the numbers t and $t+1$ are relatively prime, it follows that the product $t(t+1)$ can be a perfect square only in the case when t and $t+1$ are themselves perfect squares, that is only when $t=0$. Finally, for $t=0$ we obviously have $x=0$ and $A_y(x)=0$ for *any* y.

Thus, all the solutions of the given equation are $x=t^2$, $y=1$ and $z=t$ where t is an arbitrary integer (if the roots are understood in the arithmetical sense then we must stipulate that t can be equal to any *natural* number) and $x=0$, $y=t$ (where t is an arbitrary *natural* number) and $z=0$.

163. To solve the problem we shall use the *proof by contradiction*. Let us suppose that the equation indicated in the condition of the problem possesses integral solutions x and y only for a *finite* number of prime numbers p and that the *greatest* of the solutions is an n-digit prime number p_n. Let us form the number $x=2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot \ldots \cdot p_n$ and consider the expression $X=x^2+x+1$. Since the number $X-1=x^2+x=x(x+1)$ is divisible by all prime numbers 2, 3, 5, ..., p_n, the number X cannot be divisible by any of them. Consequently, there is a prime divisor P of the number X exceeding p_n, that is $X=Py$ where y is a natural number. (Here we do not exclude the case when $P=X$ and $y=1$.) Thus, the given equation possesses integral solutions x, y for $p=P$, which contradicts the assumption that p_n is the *greatest* value of the prime number p for which such solutions exist. This contradiction proves the assertion of the problem.

Remark. The argument used in the above solution is very similar to the one used in the well-known proof of the theorem on the existence of an infinitude of prime numbers (see the solution of Problem 349).

164. Our aim is to find the positive integral solutions of the following system of equations:

$$\left.\begin{aligned} x^2+y+z+u&=(x+v)^2\\ y^2+x+z+u&=(y+w)^2\\ z^2+x+y+u&=(z+t)^2\\ u^2+x+y+z&=(u+s)^2 \end{aligned}\right\}$$

(where x, y, z and u are the sought-for numbers). This system is equivalent to the system

$$\left.\begin{aligned} y+z+u&=2vx+v^2\\ x+z+u&=2wy+w^2\\ x+y+u&=2tz+t^2\\ x+y+z&=2su+s^2 \end{aligned}\right\} \qquad (*)$$

On adding together all equations (*) we obtain

$$(2v-3)x+(2w-3)y+(2t-3)z+(2s-3)u+v^2+$$
$$+w^2+t^2+s^2=0 \qquad (**)$$

We first of all note that equality (**) implies that at least one of the numbers $2v-3$, $2w-3$, $2t-3$ and $2s-3$ is negative: indeed, if otherwise, we should have a sum of positive numbers on the left-hand side of this equality. Now, for definiteness, let us suppose that $2v-3<0$. This is only possible when $v=0$ or $v=1$. In the former case the first equation of system (*) immediately implies that $y+z+u=0$, which is impossible when y, z and u are positive. Therefore we must assume that all the numbers v, w, t and s are positive and that $v=1$. Then equality (**) can be rewritten in the form

$$x=(2w-3)y+(2t-3)z+(2s-3)u+w^2+t^2+s^2+1 \qquad (***)$$

Let us consider the following five cases which can take place here.

1°. *The numbers x, y, z and u are all pairwise distinct.* In this case the numbers v, w, t and s are also pairwise distinct; indeed, if, for instance, we put $v=w$, then, on subtracting the first two equalities (*) from each other, we obtain $y-x=2v(x-y)$, which is impossible when v is positive and $x\neq y$. Further, under the assumption that $v=1$ the first equality (*) yields $2x=y+$ $+z+u-1$ where $x=y/2+z/2+u/2-1/2$, which contradicts equalities (***) where the coefficients in y, z and u on the right-hand side are positive integers (because w, t and s cannot be equal to 1 since they are not equal to v and $v=1$). Hence, this case is impossible.

2°. *Two of the numbers x, y, z and u are equal to each other while the others are pairwise distinct.* Here it is convenient to consider the following two subcases.

(A) $z=u$. In this case $t=s$. Equality (***) and the first equation (*) take the form

$$x=(2w-3)y+2(2t-3)z+w^2+2t^2+1$$

and

$$2x=y+2z-1$$

As before, these relations cannot hold simultaneously.

(B) $x=y$. In this case $w=v=1$. Equality (**) and the first equality (*) turn into

$$2x=(2t-3)z+(2s-3)u+t^2+s^2+2$$

and

$$x=z+u-1$$

respectively.

The substitution of the second of these equalities into the first one results in

$$(2t-5)z+(2s-5)u+t^2+s^2+4=0 \qquad (****)$$

whence it follows that at least one of the two numbers $2t-5$ and $2s-5$ must be negative. For definiteness, let $2t-5<0$; since $t>0$ and $t \neq 1$ (because $v=1$ and $t \neq v$ since $z \neq x$), it follows that $t=2$. Now, on adding the duplicated first equality (*) to the third equality (*), we obtain $4z+4x+6=4x+2z+3u$, that is $z=3u/2-3$. Let us put $t=2$ and $z=3u/2-3$ in equation (****); this yields

$$(4s-13)u+2s^2+22=0$$

It follows that $4s-13<0$. Since $s>0$, $s \neq 1$ and $s \neq 2$ we obviously have $s=3$. On substituting all these values into equations (*) we arrive at the following system of three equations of the first degree with three unknowns:

$$\left.\begin{aligned} x+z+u&=2x+1\\ 2x+u&=4z+4\\ 2x+z&=6u+9 \end{aligned}\right\}$$

From this system we easily find $x=y=96$, $z=57$ and $u=40$.

3°. *Among the numbers x, y, z and u there are two pairs of pairwise equal numbers.* For instance, let $x=y$ and $z=u$. In this case the first equation (*) yields $x=2z-1$; the substitution of this value into equation (**) results in

$$x=(2t-3)z+t^2+1$$

whence we obtain

$$(2t-5)z+t^2+2=0$$

It follows that $2t-5<0$; since $t>0$ and $t \neq 1$, this means that $t=2$. Now equations (*) reduce to the system

$$\left.\begin{aligned} x+2z&=2x+1\\ 2x+\ \ z&=4z+4 \end{aligned}\right\}$$

whence $x=y=11$, $z=u=6$.

4°. *Among the numbers x, y, z and u there are three numbers which are equal to one another.* Here it is also necessary to consider separately the following two subcases.

(*A*) $y=z=u$. In this case equation (***) and the first equation (*) take the form

$$x=3(2w-3)y+3w^2+1 \quad \text{and} \quad 2x=3y-1$$

It is evident that the last two relations cannot hold simultaneously.

(B) $x = y = z$. In this case the first equation (*) yields

$$2x + u = 2x + 1$$

whence $u = 1$; the last equation (*) results in

$$3x = 2su + s^2 = 2s + s^2 \quad \text{whence} \quad x = \frac{s(s+2)}{3}$$

Since x is an integral number, we conclude that either s or $s + 2$ is divisible by 3, that is either $s = 3k$ whence $x = k(3k + 2)$ or $s = 3k - 2$ whence $x = (3k - 2)k$; here k is an arbitrary integral number.

5°. *All the numbers x, y, z and u coincide with one another.* In this case the first equation (*) immediately yields $3x = 2x + 1$ whence $x = 1$.

Hence, we obtain the following solutions of the problem:

$$x = y = 96, \quad z = 57, \quad u = 40; \qquad x = y = 11, \quad z = u = 6;$$
$$x = y = z = k(3k \pm 2), \quad u = 1; \qquad x = y = z = u = 1$$

(the last solution corresponds to the case when we put $k = 1$ and take the sign "—" in the foregoing formulas).

165. Denoting the sought-for numbers as x and y we can write

$$x + y = xy$$

whence

$$xy - x - y + 1 = 1$$

The last relation can be written as

$$(x - 1)(y - 1) = 1$$

Since there are only two ways in which the number 1 can be factored as a product of two integral factors, we readily obtain

$$x - 1 = 1, \quad y - 1 = 1$$

whence

$$x = 2 \quad \text{and} \quad y = 2$$

or

$$x - 1 = -1, \quad y - 1 = -1$$

whence

$$x = 0 \quad \text{and} \quad y = 0$$

166. Let the numbers in question be x, y and z, then we have

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

We shall first of all show that at least one of the three numbers x, y and z must be less than 4. Indeed, if all these numbers were not smaller than 4, then the sum $1/x + 1/y + 1/z$ would be not greater than $1/4 + 1/4 + 1/4 = 3/4$. Hence, if we assume that

$x \leqslant y \leqslant z$ then x can take on only the following two values: $x = 2$ and $x = 3$ (because $x > 1$). Let us consider separately these two possibilities.

(1) $x = 2$. Then $1/y + 1/z = 1 - 1/x = 1/2$. On reducing the fractions to the common denominator and discarding the denominator we obtain

$$yz - 2y - 2z = 0, \quad \text{that is} \quad yz - 2y - 2z + 4 = 4$$

whence

$$(y-2)(z-2) = 4$$

Since y and z exceed 1, the numbers $y - 2$ and $z - 2$ cannot be negative, and therefore only the following two cases are possible:

(A) $y - 2 = 2, \quad z - 2 = 2 \quad$ whence $\quad y = 4, \quad z = 4.$

(B) $y - 2 = 1, \quad z - 2 = 4 \quad$ whence $\quad y = 3, \quad z = 6.$

(2) $x = 3; \quad$ then $\quad \frac{1}{y} + \frac{1}{z} = 1 - \frac{1}{x} = \frac{2}{3}$

whence we obtain in succession

$$2yz - 3y - 3z = 0, \; 4yz - 6y - 6z + 9 = 9 \; \text{and} \; (2y-3)(2z-3) = 9$$

Since $y \geqslant x = 3$, $2y - 3 \geqslant 3$ and $2z - 3 \geqslant 3$, only one case is possible, namely

$$2y - 3 = 3, \quad 2z - 3 = 3 \quad \text{whence} \quad y = 3 \quad \text{and} \quad z = 3$$

Hence, all the solutions of the problem are expressed by the following equalities:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1; \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1; \quad \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

167. (a) From the given equation it obviously follows that $x, y > n$; let us put $x = n + x_1$ and $y = n + y_1$. Then the equation can be rewritten in the form

$$\frac{1}{n + x_1} + \frac{1}{n + y_1} = \frac{1}{n}$$

whence

$$(n^2 + nx_1) + (n^2 + ny_1) =$$
$$= n^2 + nx_1 + ny_1 + x_1y_1, \quad \text{that is} \quad x_1y_1 = n^2 \quad (*)$$

It is clear that if n is a *prime number*, equation (*) possesses only three natural solutions, namely $(x_1, y_1) = (n, n)$, $(x_1, y_1) = (1, n^2)$ and $(x_1, y_1) = (n^2, 1)$, which lead to the following three solutions of the original equation:

$$(x, y) = (2n, 2n), \quad (x, y) = (n+1, n(n+1))$$
$$\text{and} \quad (x, y) = (n(n+1), n+1) \qquad (^{**})$$

In case $n = ab$ is a composite number, the equation may possess solutions in which x and y assume values different from those indicated by formulas (**), for instance, such is the solution corresponding to solution $(x_1, y_1) = (a^2, b^2)$ of equation (*).

(b) If $1/x + 1/y = 1/n$ then, on clearing of fractions, we obtain the equation

$$nx + ny = xy$$

which is equivalent to

$$(x-n)(y-n) = n^2$$

(cf. the solution of Problem 167 (a)). The last equation possesses $2\nu - 1$ integral solutions where ν is the number of the divisors of the number n^2 (including 1 and the number n^2 itself). To obtain all these solutions we must write down the 2ν possible systems of the form $x - n = d$, $y - n = n^2/d$ and $x - n = -d$, $y - n = -n^2/d$ (where d is a divisor of the number n^2); the system $x - n = -n$, $y - n = -n$ must not be considered because it leads to the result $x = 0$, $y = 0$ which should be discarded according to the conditions of the present problem.

If $n = 14$ then $n^2 = 196$. The divisors of the number $n^2 = 196$ are

$$1; \quad 2; \quad 4; \quad 7; \quad 14; \quad 28; \quad 49; \quad 98; \quad 196$$

Accordingly, we obtain the following 17 solutions of the equation:

x	15	16	18	21	28	42	63	112	210	13	12
y	210	112	63	42	28	21	18	16	15	−182	−84

x	10	7	−14	−35	−84	−182
y	−35	−14	7	10	12	13

(c) The given equation can be brought to the form

$$(x-z)(y-z) = z^2 \qquad (^*)$$

(cf. the solution of Problem 167 (a)). Now let t be the greatest common divisor of the three numbers x, y and z, that is $x = x_1 t$,

$y = y_1 t$ and $z = z_1 t$ where the numbers x_1, y_1 and z_1 are relatively prime. Further, let us denote by m and by n the greatest common divisors of the numbers x_1 and z_1, and of the numbers y_1 and z_1 respectively, that is let $x_1 = mx_2$, $z_1 = mz_2$ and $y_1 = ny_2$, $z_1 = nz_2'$ where x_2, z_2 and y_2, z_2' are two pairs of relatively prime numbers. The numbers m and n are relatively prime because such are x_1, y_1 and z_1. Since z_1 is divisible both by m and by n we can put $z_1 = mnp$ (that is $z_2' = mp$).

Now let us substitute $x = mx_2 t$, $y = ny_2 t$ and $z = mnpt$ into the basic equation (*). On cancelling by mnt^2 we obtain

$$(x_2 - np)(y_2 - mp) = mnp^2 \qquad (**)$$

The number x_2 is relatively prime to p because m is the *greatest* common divisor of the numbers $x_1 = mx_2$ and $z_1 = mnp$; similarly, y_2 is relatively prime to p. On opening the parentheses in equation (**) we find that the number $x_2 y_2 = x_2 mp + y_2 np$ is divisible by p. It follows that $p = 1$, and therefore the equation takes the form

$$(x_2 - n)(y_2 - m) = mn$$

The number x_2 is relatively prime to n because the three numbers $x_1 = mx_2$, $y_1 = ny_2$ and $z_1 = mn$ are relatively prime. Consequently, the number $x_2 - n$ is relatively prime to n and therefore $y_2 - m$ is divisible by n. Similarly, $x_2 - n$ is divisible by m. Thus, $x_2 - n = \pm m$, $y_2 - m = \pm n$ whence $x_2 = \pm y_2 = \pm m + n$; consequently

$$x = m(m+n)t, \quad y = \pm n(m+n)t, \quad z = mnt$$

where m, n and t are arbitrary integers.

168. (a) From the equality $x^y = y^x$ it follows that the numbers x and y have the same prime divisors:

$$x = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} \quad \text{and} \quad y = p_1^{\beta_1} p_2^{\beta_2} \ldots p_n^{\beta_n}$$

where $p_1, p_2, \ldots, p_n$ are prime numbers. Therefore from the equality $x^y = y^x$ it follows that

$$\alpha_1 y = \beta_1 x, \quad \alpha_2 y = \beta_2 x, \quad \ldots, \quad \alpha_n y = \beta_n x$$

Let us assume that $y > x$; then the equalities we have written imply

$$\alpha_1 < \beta_1, \quad \alpha_2 < \beta_2, \quad \ldots, \quad \alpha_n < \beta_n$$

Consequently, y is divisible by x, that is $y = kx$ where k is an integer. The substitution of this value of y into the equality $x^y = y^x$ results in

$$x^{kx} = (kx)^x$$

Now, on extracting the xth root from both members of this equality we obtain

$$x^k = kx, \quad \text{that is} \quad x^{k-1} = k$$

Since $y > x$ we have $k > 1$ whence $x > 1$. Further, we have $2^{2-1} = 2$, and for $x > 2$ or $k > 2$ we always have $x^{k-1} > k$. Indeed, for $k > 2$ and $x \geqslant 2$ there hold the inequalities

$$x^{k-1} \geqslant 2^{k-1} > k$$

because $2^{3-1} > 3$, and for $k = 2$, $x > 2$ we have

$$x^{k-1} = x > 2 = k$$

Therefore the given equation has a single integral solution, namely $x = 2$, $k = 2$, $y = kx = 4$.

(b) Let us denote by k the ratio y/x; then $y = kx$. The substitution of this expression of y into the given equation yields

$$x^{kx} = (kx)^x$$

On extracting the xth root from both members of the equality and dividing the result by x we obtain

$$x^{k-1} = k$$

whence

$$x = k^{\frac{1}{k-1}}, \quad y = kk^{\frac{1}{k-1}} = k^{\frac{k}{k-1}}$$

Let the rational number $1/(k-1)$ be equal to an irreducible fraction p/q. The substitution of p/q for $1/(k-1)$ into the formulas we have derived yields

$$k - 1 = \frac{q}{p}, \quad k = 1 + \frac{q}{p} = \frac{p+q}{p}, \quad \frac{k}{k-1} = \frac{p+q}{q};$$

$$x = \left(\frac{p+q}{p}\right)^{\frac{p}{q}}, \quad y = \left(\frac{p+q}{p}\right)^{\frac{p+q}{q}}$$

Since p and q are relatively prime we conclude that for x and y to be rational numbers it is necessary that the whole numbers p and $p+q$ should be equal to the qth powers of integral numbers; this is only possible when $q = 1$ because for $q \geqslant 2$ and $p = n^q$ we have the inequalities

$$n^q < p + q < (n+1)^q = n^q + qn^{q-1} + \frac{q(q-1)}{2}n^{q-2} + \ldots$$

Thus, all the positive rational numbers satisfying the given equation are expressed by the formulas

$$x = \left(\frac{p+1}{p}\right)^p, \quad y = \left(\frac{p+1}{p}\right)^{p+1}$$

where p is an arbitrary integer different from 0 and -1.

169. Let n be the number of the pupils of the 6th form and m be the number of points received by each of them. Then the number of points received by all the participants of the tournament is equal to $mn + 8$. This number is equal to the number of games played in the tournament. Since the total number of the participants of the tournament is equal to $n + 2$ and each of them played one game with each of the other $n + 1$ participants, the total number of the games played by the participants is equal to $(n + 2)(n + 1)/2$ (in the product $(n + 2)(n + 1)$ every game is taken into account twice). Therefore we obtain the equality

$$mn + 8 = \frac{(n+2)(n+1)}{2}$$

which, after simple transformations, yields

$$n(n + 3 - 2m) = 14$$

Here n is a whole number; the expression in the parentheses is also a whole number because m is either a whole number or a fraction with denominator 2.

Since n is a divisor of 14 the number n can be equal to one of the numbers 1, 2, 7 and 14. The values $n = 1$ and $n = 2$ should be discarded because in these cases the total number of the participants does not exceed 4 and hence if n were equal to 1 or 2 the two pupils of the 5th form could not receive together 8 points.

Hence, we have $n = 7$ or $n = 14$.

If $n = 7$ then $7(7 + 3 - 2m) = 14$; $m = 4$.

If $n = 14$ then $14(14 + 3 - 2m) = 14$; $m = 8$.

170. Let the number of the pupils of the 5th form be n and the number of points they received by m. Then the number of the pupils of the 6th form is $10n$ and the number of points they received is $4.5m$. The total number of the participants of the tournament is $11n$ and the number of points they receive is $5 \cdot 5m$.

The total number of points received by all the participants is equal to the number of the games they played. This number of the games is equal to $11n(11n - 1)/2$ whence

$$5.5m = \frac{11n(11n-1)}{2}$$

Consequently

$$m = n(11n - 1)$$

Each of the pupils of the 5th form played $11n - 1$ games (because the number of the participants of the tournament is equal to $11n$) and therefore the n pupils of the 5th form can receive $n(11n - 1)$ points only in the case when each of them wins all the games. This is only possible for $n = 1$ (since two pupils of the 5th form cannot simultaneously win from each other). Thus, we obtain the single solution $n = 1$, $m = 10$.

171. By the condition of the problem we have

$$\sqrt{p(p-a)(p-b)(p-c)} = 2p$$

where a, b and c are integral numbers and $p = (a+b+c)/2$. Let us denote $p - a = x$, $p - b = y$ and $p - c = z$; then we have

$$\sqrt{(x+y+z)xyz} = 2(x+y+z)$$

On squaring both members of the equality we obtain

$$xyz = 4(x+y+z)$$

Here x, y and z are either positive integers or halves of odd integers. The latter case is obviously impossible because in this case we have a fractional number on the left-hand side and an integral number on the right-hand side. Thus, x, y and z are integers.

Now let us assume that $x \geqslant y \geqslant z$. From the equation we have derived it follows that

$$x = \frac{4y+4z}{yz-4}$$

and consequently

$$\frac{4y+4z}{yz-4} \geqslant y$$

Now we can multiply the last inequality by $yz - 4$ (it is clear that $yz - 4 > 0$ because, if otherwise, x would be negative) and consider the resultant quadratic inequality with respect to y:

$$y^2z - 8y - 4z \leqslant 0, \quad \text{that is} \quad (y - y_1)(y - y_2) \leqslant 0 \qquad (*)$$

where y_1 and y_2 are the roots of the quadratic equation $zy^2 - 8y - 4z = 0$ (these roots depend on z):

$$y_1 = \frac{4+\sqrt{16+4z^2}}{z}, \qquad y_2 = \frac{4-\sqrt{16+4z^2}}{z}$$

Since y_2 is negative, we always have $y - y_2 > 0$ (because y is positive); consequently, for inequality (*) to be fulfilled it is necessary that the inequality

$$y - y_1 \leqslant 0$$

should hold, whence

$$y \leqslant \frac{4+\sqrt{16+4z^2}}{z}$$

Thus, we have $yz \leqslant 4 + \sqrt{16+4z^2}$ and therefore $z^2 - 4 \leqslant \leqslant \sqrt{16+4z^2}$ (because $z \leqslant y$). On squaring both members of the last inequality we obtain

$$z^4 - 8z^2 + 16 \leqslant 16 + 4z^2, \quad \text{that is} \quad z^4 \leqslant 12z^2$$

This relation can obviously hold only for $z \leqslant 3$.

Now let us consider in succession the cases that can take place here.

1°. $z = 1$, $y \leqslant \frac{4 + \sqrt{16 + 4}}{1} < 9$; the expression $x = \frac{4y + 4z}{yz - 4} = \frac{4y + 4}{y - 4}$ is equal to a positive integer only when $y = 5$ (in this case $x = 24$) or when $y = 6$ (in this case $x = 14$) or when $y = 8$ (in this case $x = 9$).

2°. $z = 2$, $y \leqslant \frac{4 + \sqrt{16 + 4 \cdot 4}}{2} < 5$; the expression $x = \frac{4y + 4z}{yz - 4} = \frac{4y + 8}{2y - 4} = \frac{2y + 4}{y - 2}$ is an integral number and is not less than y only when $y = 3$ (in this case $x = 10$) or when $y = 4$ (in this case $x = 6$).

3°. $z = 3$, $y \leqslant \frac{4 + \sqrt{16 + 4 \cdot 9}}{3} < 4$. For $z = y = 3$ the expression $x = \frac{4y + 4z}{yz - 4}$ is not an integral number.

Thus, we have found the following five solutions of the problem:

x	y	z	$x + y + z = p$	a	b	c
24	5	1	30	6	25	29
14	6	1	21	7	15	20
9	8	1	18	9	10	17
10	3	2	15	5	12	13
6	4	2	12	6	8	10

172. The numbers in the first row of the table can be rewritten as $0 + 1, 0 + 2, \ldots, 0 + n$ and the numbers in the last column can be rewritten as $0 + n, n + n, 2n + n, \ldots, (n - 1)n + n$. Then each of the numbers in the table is represented as a sum of two numbers the first of which is one and the same for all numbers belonging to one row and the second of which is one and the same for all numbers belonging to one column. Since the set of the chosen numbers involves one summand from every column and one summand from every row, the sum of all the first summands of the chosen numbers is equal to

$$0+n+2n+\ldots+(n-1)n=\frac{n^2(n-1)}{2}$$

and the sum of the second summands is equal to

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

Thus, the total sum S of all the chosen numbers is

$$S=\frac{n(n^2-n)}{2}+\frac{n(n+1)}{2}=\frac{n(n^2+1)}{2}$$

173. Let the number n be *odd*. If the table is symmetric about the diagonal indicated in the condition of the problem (we shall refer to it as the "principal" diagonal and denote it by the letter d) then to every number lying above d there corresponds a number lying below d which is equal to the former number; the place occupied by the latter number is symmetric about d to the place occupied by the former. It follows that the set of the numbers lying above d coincides with the set of the numbers lying below d. Therefore every number k occurs an *even* number of times a_k (it is possible that $a_k = 0$) in the set of those numbers of the table which do not belong to the diagonal d. Since every number k occurs exactly once in each of the n rows of the table (because the n places in every row are occupied by the n numbers $1, 2, \ldots, n$ arranged in some order) the total number of times the number k occurs in the given table is equal to the odd number n. Therefore the number k occurs the *odd* number of times $n - a_k$ in the diagonal d. It follows that every number k (where $1 \leqslant k \leqslant n$) occurs at least once among the numbers forming the diagonal d (because 0 is an even number), and since the total number of the places on this diagonal is equal to n, each of the numbers $1, 2, \ldots, n$ occurs exactly once in this diagonal. (In particular, it follows that $a_1 = a_2 = \ldots = a_n = n - 1$.)

The example of the table $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ shows that for an *even* number n the assertion stated in the condition of the problem *may not hold*. Using an argument similar to the above we can show that for an even n *this assertion cannot hold* (because in this case every number k occurs an *even* number of times $n - a_k$ in the set of the numbers forming the principal diagonal d).

174. Let us denote the number standing at the intersection of the ith row and the jth column as a_{ij}; the number a_{ij} can assume one of the values 1, 2, ..., n^2 ($i, j=1, \ldots, n$). Now let $1=a_{j_1j_2}$; then, by the condition of the problem, $2=a_{j_2j_3}$, $3=a_{j_3j_4}$ and so on up to $n^2=a_{j_{n^2}j_{n^2+1}}$. Here the indices $j_1, j_2, j_3, \ldots, j_{n^2+1}$ are not of course all different because each of them can take on only one of the n different values 1, 2, ..., n. It should be noted that every concrete value k occurs in the number sequence j_1, j_2, j_2, j_3, j_3, ..., j_{n^2}, j_{n^2}, j_{n^2+1} exactly $2n$ times because the given table contains n numbers in the kth row and n numbers in the kth column. Since every number "inside" the sequence $j_1, j_2, j_2, j_3, \ldots, j_{n^2}, j_{n^2+1}$ (that is every number except those in the first and in the last places) occurs exactly twice, we conclude that the fact that the value j_1 must occur in the sequence the even number of times $2n$ implies that $j_{n^2+1}=j_1$, which means that the last number of the sequence coincides with its initial number j_1.

We thus see that to solve the problem we must determine the difference between the sum of the numbers in the j_1th row of the table and the sum of the numbers in the j_1th column of the table, the indices of the row and of the column coinciding. From the rule according to which the table is formed it follows that if $a_{jj_1}=s$ (where $j \neq j_{n^2}$ and, consequently, the pair (j, j_1) is not the last one in the above sequence, that is $s \neq n^2$) then $s+1=a_{j_1j_2}$. In other words, *to each number s belonging to the j_1th column and different from n^2 there corresponds the number $s+1$ in the j_1th column.* All numbers $s+1=a_{j_1j_2}$ obtained in this way coincide with *all* numbers belonging to the j_1th row *except* only *one number* 1 (which obviously cannot follow any of the numbers of the j_1th column because it is the smallest number in the table). Therefore if we denote as $s_1, s_2, \ldots, s_{n-1}$ the numbers belonging to the j_1th column which are different from n^2, then those numbers belonging to the j_1th row which are different from 1 must be equal to $s_1+1, s_2+1, \ldots, s_{n-1}+1$. Now it follows that the difference between the sum of the numbers in the j_1th column and the sum of the numbers in the j_1th row is equal to

$$(s_1+s_2+\ldots+s_{n-1}+n^2)-$$
$$-[(s_1+1)+(s_2+1)+\ldots+(s_{n-1}+1)+1]=n^2-n$$

175. It is clear that if we denote by a_{ij} the element of the table standing at the intersection of the ith row and the jth column (where $i=1, 2, \ldots, m$ and $j=1, 2, \ldots, n$) then in all tables obtained from the original table with the aid of the "admissible" transformations this place is occupied either by the number a_{ij} or by the number $-a_{ij}$ (because the admissible transformations of the table reduce to *changes of signs* of some of the numbers contained in the table). Therefore the total number of the "admissible" tables cannot exceed 2^{mn}, that is this number is *finite*.

(The number 2^{mn} is equal to the number of all the possible sets of mn numbers a_{ij} each of which can assume *two* values.) Since the number of the tables we deal with is finite it follows that there is one table among them for which the sum of all the numbers contained in it assumes the *maximum* possible value or there are several tables with one and the same sum of the numbers which exceeds the sums of the numbers contained in all the other admissible tables. If we suppose that a row or a column of such a "maximum" table contains numbers whose sum is negative then, on changing the signs of all numbers belonging to that row or to that column we obtain a new "admissible" table for which the sum of the numbers contained in it exceeds the sum of the numbers in the former table, whence it follows that *in the "maximum" table the sum of the elements of any row and of any column must be nonnegative.*

176. Let us denote by S the sum of all numbers contained in the given table, by s_i the sum of all numbers in the ith row (where $i = 1, 2, \ldots, 100$), by σ_j the sum of all numbers in the jth column (where $j = 1, 2, \ldots, 80$) and, finally, by a_{ij} the number standing at the intersection of the ith row and the jth column. By the condition of the problem, we have

$$a_{ij} = s_i \sigma_j \qquad (*)$$

whence

$$s_i = a_{i1} + a_{i2} + \ldots + a_{i,80} = \\ = s_i\sigma_1 + s_i\sigma_2 + \ldots + s_i\sigma_{80} = s_i(\sigma_1 + \sigma_2 + \ldots + \sigma_{80}) = \\ = s_iS \qquad (i = 1, 2, \ldots, 100)$$

This means that either $S = 1$ or all the numbers s_i are equal to 0. If $s_i = 0$ for all $i = 1, \ldots, 100$ then, by virtue of (*), all the elements a_{ij} of the given table are equal to zero. However, since by the condition of the problem the "corner" element a_{11} is greater than 0, we see that $S = 1$.

177. The equality of the numbers placed in the squares symmetric about any of the two diagonals implies that among the given 64 numbers there are not more than 20 different numbers (see Fig. 17 where the numbers 1, 2, ..., 20 symbolize some numbers $a_1, a_2, \ldots, a_{20}$ among which not all must necessarily be different from one another). The numbers belonging to one of the rows, say to the ith row (where $i = 1, 2, \ldots, 8$) coincide with the numbers of the $(9-i)$th row and with the numbers of the ith and of the $(9-i)$th columns. If there is an index i such that the sum of the numbers in the ith row exceeds 518, then the same is true of the $(9-i)$th row, of the ith column and of the $(9-i)$th column (see Fig. 17 where $i = 3$ and the corresponding rows and columns are shaded). On adding together these four (equal) sums of the numbers we obtain a resultant number which cannot be less than $4\cdot 518 = 2072$. In the resultant sum each of the four numbers placed in the squares which are cross-hatched in Fig. 17 occurs twice while each of the numbers standing in the other shaded squares occurs only once. Therefore the total sum of all numbers occupying the squares which are shaded in the figure is not less than $2072 - \sigma$ where σ is the sum of the numbers written in those squares which are cross-hatched. By the condition of the problem, there must be $\sigma \leqslant 112$ (because 112 is equal to the sum of the numbers written in *all* the diagonal squares), and therefore the sum of the numbers in the shaded squares is not less than $2072 - 112 = 1960$ whereas the total sum of all numbers written on the chess-board is equal to 1956. We have thus arrived at a contradiction, which proves the assertion of the problem.

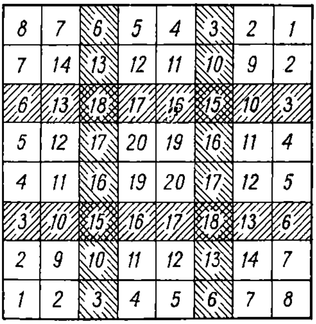


Fig. 17

178. For difiniteness, let us index the rows in the upward direction and the columns from left to right. The condition of the

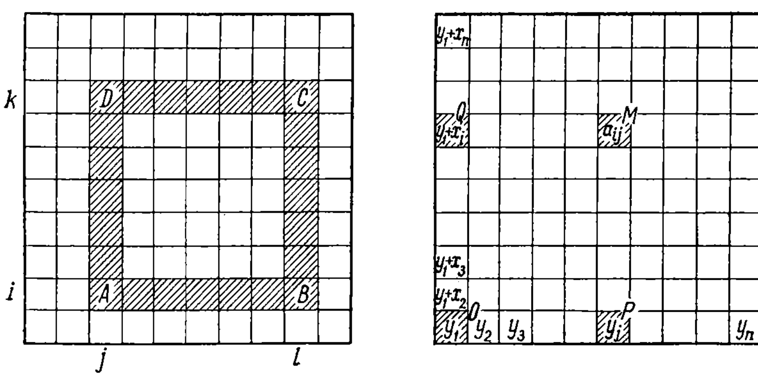


Fig. 18

problem is equivalent to the following property: *for any four indices, i, j, k and l where $i \neq k$ and $i \neq l$ there holds the equality* $a_{ij} + a_{kl} = a_{ki} + a_{jl}$. In other words, for any rectangle $ABCD$ (see Fig. 18a) on the board the sum of the numbers placed at its two opposite vertices A and C is equal to the sum of the numbers placed at the other two vertices B and D. Indeed, suppose that two of the n rooks are placed at the vertices A and C; these rooks can be of course moved to the squares B and D so that, as before, they can take a chessmen standing in the ith and in the kth rows and in the jth and in the lth columns. Hence, when the rooks are moved in this way the sum of the numbers written in the squares occupied by the rooks cannot change; that is why the sum of the numbers written in the squares A and C must necessarily be equal to the sum of the numbers written in the squares B and D.

The further course of the solution is rather simple. Let us denote by $y_1, y_2, \ldots, y_n$ the numbers written in the lowest row of the board and by $y_1, y_1+x_2, y_1+x_3, \ldots, y_1+x_n$ the numbers written in its first column (see Fig. 18*b*). Besides, let us put $x_1=0$. It is clear that for the lowest (the 1st) row and for the leftmost (the 1st) column we have $a_{1i}=y_i=x_1+y_i$ (because $x_1=0$) and $a_{j1}=x_j+y_1$. On the other hand, if $i>1$ and $j>1$ then the number a_{ij} written in a square M can be found using the square $MPOQ$ indicated in Fig. 18*b*: by what has been proved, we have

$$a_{ij}+y_1=(x_i+y_1)+y_j \quad \text{whence} \quad a_{ij}=x_i+y_j$$

It is easily seen that if there exist numbers $x_1, x_2, \ldots, x_n;$ $y_1, y_2, \ldots, y_n$ such that $a_{ij}=x_i+y_j$ then the condition indicated in the statement of the problem which is related to the arrangement of the rooks on the board (the total number of such "admissible" arrangements of the rooks is equal to n!; why?) must necessarily hold.

179. It is clear that for $j=k=i$ the equation connecting x_{ij}, x_{jk} and x_{ki} takes the form $3x_{ii}=0$; hence, $x_{ii}=0$ for all $i=1, 2, \ldots, n$. Now let $k=j\neq i$; then the equation takes the form

$$x_{ij}+x_{jj}+x_{ji}=0$$

whence, since $x_{jj}=0$ we obtain $x_{ij}=-x_{ij}$. Finally, let us add together all the equalities $x_{ij}+x_{jk}+x_{ki}=0$ corresponding to two arbitrary fixed values of i and j and to $k=1, 2, \ldots, n$; this results in

$$nx_{ij}+(x_{j1}+x_{j2}+\ldots+x_{jn})-(x_{i1}+x_{i2}+\ldots+x_{in})=0$$

(here we have used the relation $x_{ki}=-x_{ik}$). Let us denote

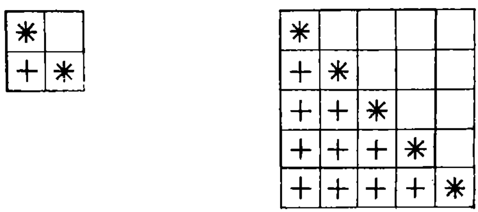
$$\frac{1}{n}(x_{i1}+x_{i2}+\ldots+x_{in})=t_i, \quad i=1, 2, \ldots, n$$

It follows that

$$x_{ij}=t_i-t_j$$

180. It is clear that the assertion stated in the problem holds for a "board" of dimension 1×1 (consisting of one single square;

in this case the assertion is trivial because the "board" contains a single square in which the star is placed). This makes it possible to use the *method of mathematical induction.* For $n = 2$ the board obviously contains a column (and, consequently, a row as well) in which there is exactly one star. On interchanging (if necessary) the rows and the columns so that the empty square occupies the rightmost upper place we obtain a table having a "triangular" structure (see Fig. 19*a* where the sign "+" indicates that in the corresponding square the star may or may not stand). Let us prove that *for an arbitrary n as well the rows and the columns of the table can be interchanged in such a way that the*



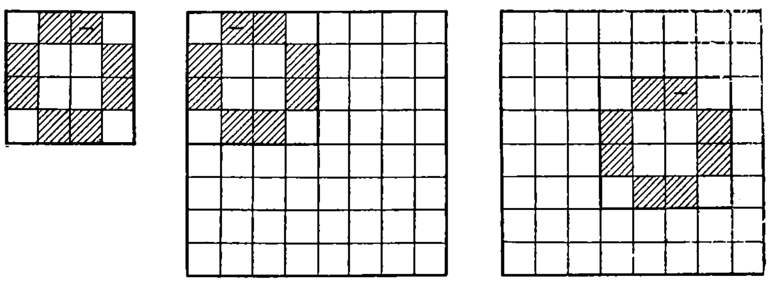
(a) (b)

Fig. 19

table takes a "triangular" form in which all the stars in the table are placed along the diagonal joining the left upper corner of the table and the right lower corner and, perhaps, below that diagonal while all the places above the diagonal are empty. Indeed, let us suppose that this assertion has already been proved for all tables of dimension $(n-1)\times(n-1)$ satisfying the required conditions and let us consider a table of dimension $n \times n$ satisfying these conditions. The latter table contains a column in which there is exactly one star. Let us move this column to the last place and then interchange the rows so that the star occupies the lowest place in that column (see Fig. 19*b*). The resultant table of dimension $n \times n$ contains a "sub-table" of dimension $(n-1)\times(n-1)$ which also satisfies the conditions of the problem. By the hypothesis, the rows and the columns of this sub-table can be interchanged so that it takes the "triangular" form; after the rows and the columns are interchanged in this way the original table of dimension $n \times n$ also assumes the "triangular" form.

The "triangular" structure of the tables we deal with (we can limit ourselves to the tables having this "triangular" structure because the properties of the tables we are interested in are preserved when the rows and the columns are interchanged in an arbitrary manner) proves the equivalence of the rows and the columns of the tables, whence it follows that the assertion of the problem (which is quite obvious for the "triangular" tables) holds in the general case as well.

181. (a) On a board of dimension 4×4 there are 4 columns, 4 rows and $2 \cdot 7 = 14$ "inclined lines" of squares parallel to the diagonals of the table (among them there are 4 "inclined lines" consisting of one corner square). It can readily be seen that each of these $4 + 4 + 14 = 22$ vertical, horizontal and inclined lines of squares contains an even number of squares (more precisely, this number is equal either to 0 or to 2) which are shaded in Fig. 20*a*. By the condition of the problem, originally only one of



(a) *(b)* *(c)*

Fig. 20

these squares contains the sign "—"; therefore under all the admissible changes of the signs the number of shaded squares marked with the sign "—" remains *odd* and hence it can never become equal to 0.

(b) For any location of the sign "—" on the board we can always "cut out" of the given board of dimension 8×8 a "smaller" board of dimension 4×4 so that the arrangement of the signs on the latter board is as was indicated in the condition of Problem 181 (a) (see Figs. 20, *b*, *c* where two possible variants of the location of the sign "—" on the larger board are shown). Since the admissible changes of the signs on the larger board generate the corresponding changes of the signs on the "smaller" board which satisfy the conditions of Problem 181 (a), the required proof follows from the result established in Problem 181 (a).

182. **(a)** First of all let us find the number of the possible quadratic arrays of squares of dimension 3×3 and of dimension 4×4 which can be placed on the chess-board. It is clear that the left lower corner of a quadratic array of squares of dimension

3×3 may coincide with any square belonging to the quadratic array of dimension 6×6 which is shaded in Fig. 21 by horizontal lines and that the left lower corner of a quadratic array of squares of dimension 4×4 may coincide with any square of the (smaller) quadratic array of dimension 5×5 which is shaded in the figure by vertical lines. Thus, the total number of quadratic arrays of squares of dimension 3×3 and of dimension 4×4 that can be placed on the board is equal to $6 \cdot 6 + 5 \cdot 5 = 36 + 25 = 61$. Consequently, starting with the board whose all squares contain the signs "+" and performing all the admissible operations described in the condition of the problem we can obtain not more than 2^{61} possible arrangements of the signs in all the squares of the board because in each of the 61 quadratic arrays of dimensions 3×3 and 4×4 we may or may not change all the signs independently of the other squares of the table. Since the total number of the possible arrangements of the signs "+" and "—" in the 64 squares of the chess-board is equal to $2^{64} > 2^{61}$, it is *impossible* to obtain *all the possible* arrangements of the signs on the board starting with the arrangement in which all the squares contain only the signs "+". Further, if we take an arrangement of the signs "+" and "—" which cannot be obtained in the way described above from the arrangement involving only the signs "+" then, conversely, starting with the former arrangement of the signs we can never arrive at the arrangement involving only the signs "+". It follows that the answer to the question posed in the problem is *negative*.

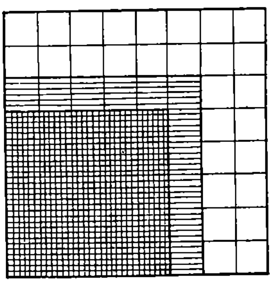


Fig. 21

(b) This problem is very close to Problem 182 (a). The total number of the quadratic arrays of squares of dimension 2×2 which can be taken on the chess-board is equal to $7 \cdot 7 = 49$ and the number of pairs of neighbouring rows and of neighbouring columns of the board is equal to $7 + 7 = 14$; hence, there exist $10^{49+14} = 10^{63}$ ways in which the last digits of the numbers placed in the squares of the board can be changed. (Among these 10^{63} ways there is one under which none of the digits is changed.) It follows that starting with an arrangement of numbers such that all the numbers end with 0 we cannot obtain *all the possible* arrangements of the last digits of the numbers placed in the 64 squares of the board because the number of such arrangements is equal to $10^{64} > 10^{63}$. Consequently, there are such arrangements of the last digits from which it is *impossible* to pass to the case when all the last digits are equal to 0.

Remark. It is clear that this solution of the problem and the negative answer to the question stated in the problem remain valid in the general case when it is required to achieve the divisibility of all the numbers in the table by any natural number n, say by the number 1976, instead of the divisibility by 10.

183. (a) The answer to the question is positive. It is clear that we can arrive at the case when at least one of the three sets contains *exactly one* ball by performing a certain number of times the operation of taking simultaneously one ball from each of the three sets of balls. Next we duplicate the number of the balls in the set (or in the sets) containing 1 ball and then again take simultaneously one ball from each of the three sets. This results in a decrease by 1 of the numbers of the balls in those sets for which these numbers are different from 1 while those sets each of which contains exactly one ball retain their numbers of balls (equal to 1). Proceeding in this way we can arrive at the case when each of the sets contains *exactly one* ball, after which all the remaining balls can be taken.

(b) This problem is quite similar to Problem 183 (a) (it should be noted that the numbers of balls dealt with in Problem 183 (a) can be arranged as a "table" consisting of one row and three columns in which every column contains one number). First let us consider the numbers belonging to only one row, say to the first. Then the admissible operations allow us to duplicate any of the numbers in that row or to subtract unity from all these numbers. Therefore, in exactly the same manner as in the solution of Problem 183 (a), we can make all the numbers in this row turn into 0. After this, in just the same way, we can make all the numbers of the second row turn into 0 and then perform the same operations on all the other rows of the table.

184. Let us write the number a in the "binary number system" that is in the form

$$a = \alpha_n \cdot 2^n + \alpha_{n-1} \cdot 2^{n-1} + \alpha_{n-2} \cdot 2^{n-2} + \ldots + \alpha_1 \cdot 2 + \alpha_0 \cdot 1$$

where each of the "digits" $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n$ is equal to 0 or to 1 (we can of course assume that the digit α_n of the number a is equal to 1). It is clear that if a is an *even* number then $\alpha_0 = 0$ and $a_1 = a/2 = \alpha_n \cdot 2^{n-1} + \alpha_{n-1} \cdot 2^{n-2} + \alpha_{n-2} \cdot 2^{n-3} + \ldots + \alpha_1 \cdot 1$. If a is an *odd* number, we have $\alpha_0 = 1$, and in this case the number $a_1 = (a-1)/2$ has the above structure. Thus, if the binary representation of the number a is written as a sequence of "digits" 0 and 1 in the form $a = \overline{\alpha_n\alpha_{n-1}\alpha_{n-2}\ldots\alpha_1\alpha_0}$ then the binary representation of a_1 has the form $a_1 = \overline{\alpha_n\alpha_{n-1}\alpha_{n-2}\ldots\alpha_1}$. Accordingly, the number a_2 is written as $a_2 = \overline{\alpha_n\alpha_{n-1}\ldots\alpha_2}$, the number a_3 as $a_3 = \overline{\alpha_n\alpha_{n-1}\ldots\alpha_3}$ and so on up to the number $a_n = \alpha_n = 1$

inclusive. On the other hand, we obviously have $b_1 = 2b$, $b_2 = 2b_1 = 2^2b$, $b_3 = 2^3b$ and so on up to the number $b_n = 2^nb$.

It is evident that the number $a_i = \overline{\alpha_n\alpha_{n-1}\ldots\alpha_i} = \alpha_n \cdot 2^{n-i} + \alpha_{n-1} \cdot 2^{n-i-1} + \ldots + \alpha_{i+1} \cdot 2 + \alpha_i \cdot 1$ (where $i = 0, 1, 2, \ldots, n$) is *odd* when the digit α_i is equal to 1 (we remind the reader that α_i can only be equal to 0 or 1). Thus, in this case we have $b_i = 2^ib = (\alpha_i \cdot 2^i)b$. Therefore the sum we are interested in which involves all the numbers b_i corresponding to the *odd* numbers a_i can be written in the form

$$[\alpha_n \cdot 2^n + \alpha_{n-1} \cdot 2^{n-1} + \ldots + \alpha_1 \cdot 2 + \alpha_0 \cdot 1]b = \overline{\alpha_n\alpha_{n-1}\ldots\alpha_1\alpha_0} \cdot b = ab$$

where only the summands corresponding to the values of the "digits" $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ equal to 1 give real inputs to the sum $\alpha_n \cdot 2^n + \alpha_{n-1} \cdot 2^{n-1} + \ldots + \alpha_1 \cdot 2 + \alpha_0 \cdot 1 = a$.

185. A concise solution of the problem can be obtained by using the *method of mathematical induction*. Let us agree to denote by u_k the kth Fibonacci number where $k = 1, 2, 3, \ldots$. Let us suppose that the assertion of the problem has already been proved for all natural numbers n smaller than the kth Fibonacci number u_k (by the way, the validity of the assertion for all numbers smaller than $u_5 = 5$ can be verified directly). It is clear that the same assertion will hold for the number u_k itself as well. Further, since all the numbers lying between u_k and $u_{k+1} = u_k + u_{k-1}$ can be represented in the form $u_k + m$ where $0 < m < u_{k-1}$ and since, by the induction hypothesis, every number m smaller than u_{k-1} can be represented in the form of a sum of some different Fibonacci's numbers whose indices are less than $k - 1$, the number $n = u_k + m$ can also be represented as a sum of Fibonacci's numbers (among which the greatest number is equal to u_k while the other numbers have indices less than $k - 1$). Thus, we have shown that the assertion also holds for all natural numbers smaller than u_{k+1}, whence it follows that it is true for *all* the natural numbers.

Remark. What has been proved implies that the sums of Fibonacci's numbers we have considered (in these sums we of course put $2 = u_3$ but not $2 = 1 + 1 = u_1 + u_2$ and $4 = 3 + 1 = u_4 + u_2$ but not $4 = 2 + 1 + 1 = u_3 + u_2 + u_1$ and the like) *cannot involve two "neighbouring" Fibonacci's numbers* (why?). It can easily be shown that *the set consisting of all possible sums of Fibonacci's numbers satisfying the last condition is nothing but the set of all natural numbers* and that every number occurs in the set of all such sums *exactly once.*

186. We are interested in the sums $s_k = u_{k+1} + u_{k+2} + \ldots + u_{k+8}$ of eight consecutive Fibonacci numbers. Since Fibonacci's numbers obviously form an increasing sequence (that is $u_1 = u_2 < u_3 < u_4 < \ldots < u_n < u_{n+1} < \ldots$), it is clear that to prove the assertion of the problem it is sufficient to show that the sum s_k lies between u_{k+9} and u_{k+10}, that is $u_{k+9} < s_k < u_{k+10}$. It is obvious that

$$u_{k+9} = u_{k+8} + u_{k+7} < u_{k+8} + u_{k+7} + u_{k+6} + \dots + u_{k+1} = s_k$$

and hence it only remains to prove the inequality $s_k < u_{k+10}$. It can easily be seen that *the sum $S_n = u_1 + u_2 + u_3 + \dots + u_n$ of the first n Fibonacci numbers is smaller by unity than the $(n+2)$th Fibonacci number u_{n+2}.* For instance, this can be proved with the aid of the *method of mathematical induction.* Indeed, we obviously have $S_2 = 1 + 1 = 3 - 1 = u_4 - 1$. On the other hand, if the assertion we have just stated holds for an index n, then on replacing n by $n+1$ and using the induction hypothesis we obtain

$$S_{n+1} = u_1 + u_2 + \dots + u_n + u_{n+1} = S_n + u_{n+1} = \\ = (u_{n+2} - 1) + u_{n+1} = (u_{n+1} + u_{n+2}) - 1 = u_{n+3} - 1$$

which is what we intended to prove. Now we can write

$$s_k = u_{k+1} + u_{k+2} + \dots + u_{k+8} = S_{k+8} - S_k = \\ = (u_{k+10} - 1) - (u_{k+2} - 1) = u_{k+10} - u_{k+2} < u_{k+10}$$

which completes the solution of the problem.

Remark 1. Evidently, in just the same way we can prove that *a sum of any m consecutive Fibonacci numbers cannot be equal to a Fibonacci number for* $m \geqslant 3$.

Remark 2. The inequality $s_k < u_{k+10}$ can also be proved by using the following consecutive transformations of the expression for u_{k+10}:

$$u_{k+10} = u_{k+9} + u_{k+8} = (u_{k+8} + u_{k+7}) + u_{k+8} = \\ = u_{k+8} + u_{k+7} + (u_{k+7} + u_{k+6}) = u_{k+8} + u_{k+7} + u_{k+6} + (u_{k+6} + u_{k+5}) = \\ = u_{k+8} + u_{k+7} + u_{k+6} + u_{k+5} + (u_{k+5} + u_{k+4}) = \dots \\ \dots = u_{k+8} + u_{k+7} + \dots + u_{k+2} + (u_{k+2} + u_{k+1}) = s_k + u_{k+2} > s_k$$

187. Let us agree to denote as $\alpha_1, \alpha_2, \alpha_3, \dots$ the remainders resulting from the division by 5 of the Fibonacci numbers $u_1, u_2, u_3, \dots$. It is obvious that from the equality $u_k = u_{k-1} + u_{k-2}$ where $k = 3, 4, 5, \dots$ it follows that

$$\alpha_k = \begin{cases} \alpha_{k-1} + \alpha_{k-2} & \text{for } \alpha_{k-1} + \alpha_{k-2} < 5 \\ \alpha_{k-1} + \alpha_{k-2} - 5 & \text{for } \alpha_{k-1} + \alpha_{k-2} \geqslant 5 \end{cases} \qquad (*)$$

Formulas (*) make it possible to find any number of terms of the sequence α_k:

$$\underbrace{1;\ 1;\ 2;\ 3;\ 0;\ 3;\ 3;\ 1;\ 4;\ 0;\ 4;\ 4;\ 3;\ 2;\ 0;\ 2;\ 2;\ 4;\ 1;\ 0;}_{20 \text{ numbers}}\ 1;\ 1;\ \dots$$

It is seen that the computation of any term of the sequence α_k following the first 20 terms can be performed without using formulas (*) because $\alpha_{21} = \alpha_1$ and $\alpha_{22} = \alpha_2$, and consequently, by virtue of (*), $\alpha_{23} = \alpha_3$, $\alpha_{24} = \alpha_4$ and so on, that is the "sequence of the remainders" $\alpha_1, \alpha_2, \alpha_3, \ldots$ is *periodic*, the period consisting of 20 numbers. Since the first group of 20 remainders α_k has zeros at the 5th, 10th, 15th and 20th places, the same is true of all the other places whose indices are multiple of 5.

188. Let us leave only the last four digits (and discard the other digits) in each member of the Fibonacci sequence which is written with the aid of five or more digits. This results in a number sequence whose every member is smaller than 10^4. Let us denote by a_k the member of this sequence occupying the kth place. Note that if the numbers a_{k+1} and a_k are known, it is possible to find a_{k-1} because the $(k-1)$th member of the Fibonacci sequence is equal to the difference between its $(k+1)$th and kth members, and the last four digits of the difference can be determined from the last four digits of the minuend and subtrahend. It follows that if we have $a_k = a_{n+k}$ and $a_{k+1} = a_{n+k+1}$ for some indices k and n then $a_{k-1} = a_{n+k-1}$, $a_{k-2} = a_{n+k-2}, \ldots, a_1 = a_{n+1}$. Since $a_1 = 0$ we conclude that $a_{n+1} = 0$, i.e. that the number occupying the $(n+1)$th place in the Fibonacci sequence ends with four noughts.

It remains to show that among the $10^8 + 1$ pairs of numbers

$$\begin{array}{ll} a_1, & a_2 \\ a_2, & a_3 \\ \cdots & \cdots \\ a_{10^8}, & a_{10^8+1} \\ a_{10^8+1}, & a_{10^8+2} \end{array}$$

there are at least two coincident pairs. But this is quite evident because, on the one hand, each of the numbers $a_1, a_2, a_3, \ldots, a_{10^8+2}$ does not exceed 10^4, and, on the other hand, using the 10^4 numbers 0; 1; 2; 3; 4; ...; 9999 we can form only $10^4 \cdot 10^4 = 10^8$ different pairs of numbers (since the first number in a pair can assume 10^4 different values and the other number can also assume 10^4 different values).

Remark. It is even possible to indicate exactly the first number among the members of Fibonacci's sequence which has four noughts at the end of its decimal representation; the index of this number is equal to 7501.

189. Since for $n > 1$, we have

$$a_n^2 = \left(a_{n-1} + \frac{1}{a_{n-1}}\right)^2 = a_{n-1}^2 + 2 + \frac{1}{a_{n-1}^2} > a_{n-1}^2 + 2$$

and therefore the equality $a_1^2 = 1$ implies

$$a_2^2 > 1 + 2 = 3; \quad a_3^2 > 1 + 2 \cdot 2 = 5; \quad a_4^2 > 1 + 3 \cdot 2 = 7; \ \ldots$$
$$\ldots; \ a_{100}^2 > 1 + 99 \cdot 2 = 199$$

whence it follows that $a_{100} > \sqrt{199} > 14$. Similarly, using the inequality

$$a_n^2 = \left(a_{n-1} + \frac{1}{a_{n-1}}\right)^2 = a_{n-1}^2 + 2 + \frac{1}{a_{n-1}^2} \leqslant a_{n-1}^2 + 3$$

which also holds for all $n > 1$ (it turns into equality only for $n = 2$) we obtain

$$a_2^2 = 1 + 3 = 4; \quad a_3^2 < 1 + 2 \cdot 3 = 7; \quad a_4^2 < 1 + 3 \cdot 3 = 10; \ \ldots$$
$$\ldots; \ a_{100}^2 < 1 + 99 \cdot 3 = 298$$

whence it follows that $a_{100} < \sqrt{298} < 18$.

190. *First solution.* Let us add to the given sequence one more number a_{n+1} such that $|a_{n+1}| = |a_n + 1|$ and then square all the equalities given in the condition of the problem (including the "additional" equality connecting the numbers a_n and a_{n+1}):

$$a_1^2 = 0; \quad a_2^2 = (a_1 + 1)^2 = a_1^2 + 2a_1 + 1;$$
$$a_3^2 = (a_2 + 1)^2 = a_2^2 + 2a_2 + 1; \ \ldots; \quad a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1;$$
$$a_{n+1}^2 = a_n^2 + 2a_n + 1$$

On adding together all these relations we obtain

$$a_1^2 + a_2^2 + a_3^2 + \ \ldots \ + a_n^2 + a_{n+1}^2 = 0 + a_1^2 + a_2^2 + \ \ldots$$
$$\ldots \ + a_{n-1}^2 + a_n^2 + 2(a_1 + a_2 + a_3 + \ \ldots \ + a_n) + n \cdot 1$$

The last equality implies

$$2(a_1 + a_2 + a_3 + \ \ldots \ + a_n) = -n + a_{n+1}^2 \geqslant -n$$

and consequently

$$\frac{a_1 + a_2 + a_3 + \ \ldots \ + a_n}{n} \geqslant -\frac{1}{2}$$

Second solution. For $n = 1$ the arithmetic mean under consideration is equal to $a_1/1 = 0$; for $n = 2$ (in this case we obviously have $a_2^2 = 1$, and hence a_2 is equal to $+1$ or to -1) it is equal to $(a_1 + a_2)/2 = (0 + a_2)/2 = a_2/2$, that is it is equal to $1/2$ or to $-1/2$; thus, in these two cases the required inequality is fulfilled. Now let us use the *method of mathematical induction*; to this end we assume that in the case when the number of the members of the given sequence is less than n the assertion stated in the problem holds and then prove that under this assumption it holds for the case of n members as well. Let $a_1, a_2, \ldots, a_n$ be a sequence of n numbers satisfying the conditions of the problem and let a_m (where the index m takes one of the values $1, 2, 3, \ldots, n$) be the *smallest* of these numbers. In this case we can of course assume that $a_m < 0$ (if all the given numbers are nonnegative no proof is needed because the arithmetic mean of such numbers exceeds $-1/2$) and that $a_m = -a_{m-1} - 1$ (because only in this case we have $|a_m| = |a_{m-1} + 1|$ and $a_m < a_{m-1}$). This means that we can assume that the arithmetic mean of the numbers a_{m-1} and a_m is equal to $-1/2$:

$$\frac{a_{m-1} + a_m}{2} = -\frac{1}{2}$$

Now it should be noted that *if we exclude the numbers a_{m-1} and a_m from the given sequence the remaining $m - 2$ numbers also form a sequence satisfying the required conditions.* Indeed, it is clear that $m \neq 1$ (because $a_1 = 0$ and $a_m < 0$); this means that the number a_{m-1} makes sense. Further, if $m = 2$ then $a_m = a_2 = -1$ and $a_3 = 0$ (because $|a_3| = |a_2 + 1|$), and therefore we can discard the numbers a_1 and a_2 and consider the sequence as starting with the number $a_3 = 0$. Similarly, in case $m = n$ we can discard the last two terms a_{n-1} and a_n, the remaining numbers satisfying the required equalities. Finally, if $3 \leqslant m \leqslant n - 1$ then

$$|a_{m-2} + 1| = |a_{m-1}| = |-a_m - 1| \text{ and } |a_{m+1}| = |a_m + 1| = |-a_m - 1|$$

whence it follows that $|a_{m-2} + 1| = |a_{m+1}|$; this means that the numbers $a_1, a_2, \ldots, a_{m-2}, a_{m+1}, \ldots, a_n$ do in fact satisfy the required conditions stated in the problem.

By the induction hypothesis which holds for $n - 2$ numbers we have

$$\frac{a_1 + a_2 + \ldots + a_{m-2} + a_{m+1} + \ldots + a_n}{n - 2} \geqslant -\frac{1}{2}$$

that is $a_1 + a_2 + \ldots + a_{m-2} + a_{m+1} + \ldots + a_n \geqslant -(n-2)/2$. On the other hand, as we already know, there must be

$$\frac{a_{m-1} + a_m}{2} = -\frac{1}{2}, \quad \text{that is} \quad a_{m-1} + a_m = -1$$

and therefore

$$a_1 + a_2 + \ldots + a_{m-2} + a_{m-1} + a_m + a_{m+1} + \ldots$$
$$\ldots + a_n \geqslant -\frac{1}{2}(n-2) + (-1) = -\frac{1}{2}n$$

We thus see that $(a_1 + a_2 + \ldots + a_n)/n \geqslant -1/2$ which is what we intended to prove.

191. It is quite clear that only one of the first two numbers a_0 and a_1 can be the greatest member of the given sequence (because for $k \geqslant 2$ every a_k must necessarily be less than max $[a_{k-1}, a_{k-2}]$ *. It can also be easily seen that in the sequence of maximum length *the number* a_1 *must be the greatest one* because if a sequence starts with numbers $a_1, a_2, a_3, \ldots$ where $a_1 > a_2$ then we can "continue the sequence to the left" without changing its properties by writing it as $a_0 = a_1 - a_2$, a_1, $a_2 = |a_1 - a_0|$, $a_3, \ldots$ (here we obviously have $a_1 > a_0 = a_1 - a_2$). Therefore in what follows we shall limit ourselves to the investigation of the sequences whose first member a_1 *is the greatest one* (however, it should be taken into account that when stating the final answer to the problem we should increase by 1 the length of the sequence because of the presence of the number $a_0 = a_1 - a_2$ having the "zeroth" index).

It is evident that if the greatest member of a sequence is $a_1 = 1$, this sequence consists of not more than two numbers (it can be continued by adding only one number $a_2 = 1$). If the greatest member of a sequence is $a_1 = 2$, the sequence contains not more than three members (in case $a_2 = 2$ the sequence has the form 2, 2 and in case $a_2 = 1$ it has the form 2, 1, 1). If $a_1 = 3$ the sequence contains not more than five members (if we put $a_2 = 3$ or $a_2 = 2$ or $a_2 = 1$ we arrive at the sequences 3, 3 or 3, 2, 1 or 3, 1, 2, 1, 1 respectively). These examples hint that the "optimal" sequence probably begins with the numbers $a_1 = n$ and $a_2 = 1$, its initial part being of the form

$$n;\ 1;\ n-1;\ n-2;\ 1;\ \ldots \tag{*}$$

It follows that on denoting the number of the members forming sequence (*) by k_n we can write

$$k_n = 3 + k_{n-2} \tag{**}$$

(because, starting with the 4th member $a_4 = n - 2$, we arrive at a similar sequence for which n is replaced by $n - 2$).

From relation (**) we find

$$k_1 = 2,\quad k_2 = 3,\quad k_3 = 3 + 2 = 5,\quad k_4 = 3 + 3 = 6,$$
$$k_5 = 3 + 5 = 8,\quad k_6 = 3 + 6 = 9,\ \ldots$$

All these numbers k_n can be described by the formula

$$k_n = \left[\frac{3n+1}{2}\right] \tag{***}$$

* Cf. the footnote on page 265.

where, as usual (cf, page 36), the square brackets designate the *integral part* of a number. Formula (***) can obviously be simply derived from (**) with the aid of the *method of mathematical induction*. Indeed, as was shown, it holds for $n = 1$ and $n = 2$; if (***) holds for a value $n-2$ then for n this relation also holds because

$$k_n = k_{n-2} + 3 = \left[\frac{3(n-2)+1}{2}\right] + 3 = \left[\frac{3n+1}{2}\right]$$

Thus, we have already constructed sequence (*) (which starts with its greatest number) of length k_n which is connected with the magnitude n of the greatest number by relation (***). Now we shall show that if *the sequence described in the conditions of the problem starts with the greatest number $a_1 = n$ then its length cannot exceed the number k_n defined by formula* (***). To carry out the proof it is natural to use the *induction method*. Let us assume that the proposition we have stated has already been proved for all n smaller than a certain value (the fact that this proposition is true for $n = 1$, $n = 2$ and $n = 3$ was already checked) and then show that under this assumption the proposition is true for that value as well. Indeed, suppose that we are given a sequence satisfying the conditions of the problem which starts with the numbers

$$n, m, \ldots$$

where $m \leqslant n$. Next let us consider the possible variants (corresponding to different values of m) that can take place here.

1°. If $m = n$ the sequence ends with the second term; the proposition we have stated obviously holds for such a sequence.

2°. If n is an even number and $m = n/2$ then the sequence ends with the third term, namely it has the form $n, n/2, n/2$; in this case the length of the sequence obviously does not exceed k_n either.

3°. If $n > m > n/2$ then after the first member n of the sequence has been discarded, we obtain the sequence $m, n-m, \ldots$ which also starts with its greatest member m. By the induction hypothesis, the remaining sequence contains not more than $k_m = [(3m+1)/2]$ members. Since we have discarded one member and since $m \leqslant n-1$, the number of the members of the entire sequence is not more than

$$1 + \left[\frac{3m+1}{2}\right] \leqslant 1 + \left[\frac{3(n-1)+1}{2}\right] = \left[\frac{3n}{2}\right] \leqslant \left[\frac{3n+1}{2}\right] = k_n$$

4°. Finally, if, $m < n/2$ then the beginning of the sequence is of the form $n, m, n-m, n-2m, m, \ldots$ If in this case we have

$n-2m \geqslant m$ (that is if $m \leqslant n/3$) then the sequence obtained after the first three members of the given sequence have been discarded (this resultant sequence starts with the number $n-2m$) is such that its greatest member is the first one; therefore, according to the induction hypothesis, the number of terms in the resultant sequence does not exceed

$$k_{n-2m}=\left[\frac{3(n-2m)+1}{2}\right]\leqslant\left[\frac{3(n-2)+1}{2}\right]=\left[\frac{3n+1}{2}\right]-3$$

(because $m \geqslant 1$), and consequently the total number of the members of the given sequence does not exceed $[(3n+1)/2]=k_n$. If $n-2m<m$ (that is $m>n/3$) then the sixth member of the sequence is equal to $m-(n-2m)$ and hence it is less than m. Therefore in this case the sequence obtained after the first four terms have been discarded is such that its greatest member m stands at the beginning of the sequence and hence the number of the terms of the resultant sequence does not exceed $k_m=[(3m+1)/2]\leqslant$ $\leqslant[(3n+2)/4]$. Consequently, the total number of terms of the given sequence is again not greater than $k_m+4\leqslant[(3n+2)/4]+$ $+4\leqslant[(3n+1)/2]=k_n$.

We have thus completed the proof of the assertion for the sequences beginning with their greatest members. It follows that the sequence described in the condition of the problem cannot contain more than $1+k_{1967}=1+[(3\cdot 1967+1)/2]=2952$ members; the number of the members of the sequence is equal to 2952 if and only if the sequence begins with the numbers 1966; 1967; 1; 1966; 1965; 1; 1964; 1963; 1; ... .

192. In this problem we shall use the *proof by contradiction*. Let us suppose that the sequence α_1, $\overline{\alpha_1\alpha_2}$, $\overline{\alpha_1\alpha_2\alpha_3}$, ... contains only a *finite* number of composite numbers. It is clear that in this case the sequence α_1, α_2, α_3, ... contains only a finite number of *even* digits (because every number whose decimal representation ends with an even digit is a composite number), and consequently all the digits α_n, α_{n+1}, α_{n+2}, ... (beginning with the nth digit where n is some index) must be *odd*. In just the same way we can show that in the given sequence α_1, α_2, α_3, ... of digits there are only a finite number of *fives* (because every number having 5 as its last digit is divisible by 5). Thus, beginning with some place in the sequence, the digits following this place can only be *ones*, *threes* and *sevens* (because, by the condition of the problem, this sequence of digits does not contain nines). Further, when 3 is additionally written at the end of a number the remainder resulting from the division of the number by 3 does not change and when one or seven is additionally written at the end of a number the remainder resulting from the division of the number by 3 increases by 1 (because $7 = 2\cdot 3 + 1$). Therefore, if the number of ones and sevens in the sequence of digits is infinite then every third number ending with 1 or with 7 is divisible by 3, that is it is a composite number. Thus, for the sequence α_1, $\overline{\alpha_1\alpha_2}$, $\overline{\alpha_1\alpha_2\alpha_3}$, ... to possess the property that only a finite number of its members are composite numbers we must additionally write at the end of the given numbers the digit 3 beginning with the Nth place in the sequence of digits $\alpha_1, \alpha_2, \alpha_3, \ldots$ where N is a certain value of the index. In this way we arrive at a sequence of numbers having the form

$$M = \overline{\alpha_1\alpha_2 \ldots \alpha_{N-1}\underbrace{333 \ldots 3}_{k \text{ times}}} = 10^k A + 3B$$

where

$$A = \overline{\alpha_1\alpha_2 \ldots \alpha_{N-1}} \quad \text{and} \quad B = \underbrace{111 \ldots 1}_{k \text{ times}} \qquad (k = 1, 2, 3, \ldots)$$

Now let p be a prime divisor of the number A (it is possible that p coincides with A). We can assume that p is different from 2 and from 5 because if N is sufficiently large the digit α_{N-1} is equal to 1 or to 7 or to 3 (the case $\alpha_{N-1} = 3$ is not excluded here). Further, there are *infinitely many* values of k such that *the number B written with the aid of k ones is divisible by p* (see the remark at the end of the solution of Problem 144). To all these values of k there correspond composite numbers $M = 10^k A + 3B$ (which are divisible by A). This contradicts the assumption that among the numbers M there are infinitely many composite numbers, which proves the assertion stated in the problem.

193. (a) The last digit of a sum of four numbers and the sum itself are simultaneously even or odd depending solely on whether the digits in the given sequence are even or odd. Let us agree to symbolize by the letter *o* an odd number and by the letter *e* an even number. It is readily seen that the beginning of the given sequence can be written symbolically in the form

oooeooooeooooe ...

and that this sequence continues *periodically*: after every four-tuple of odd numbers *o* there appears one even number *e*. It follows that the sequence 1234 having the structure *oeoe* cannot occur in the given sequence of digits.

(b) The number of different four-tuples of digits each of which assumes the ten possible values 0, 1, ..., 9, is equal to 10^4. Therefore in the sequence of 10 004 digits *one and the same* four-tuple of digits (standing side by side) must necessarily occur twice.

Further, if the ith, the $(i+1)$th, the $(i+2)$th and the $(i+3)$th digits coincide with the jth, the $(j+1)$th, the $(j+2)$th and the $(j+3)$th digits respectively (where we assume that $j < i$) then, by virtue of the rule according to which the given sequence is formed, the $(i-1)$th digit coincides with the $(j-1)$th digit, the $(i-2)$th digit coincides with the $(j-2)$th digit etc. Consequently, the $(i-j-1)$th, the $(i-j)$th, the $(i-j+1)$th and the $(i-j+2)$th digits of the given sequence must necessarily coincide with the 1st, the 2nd, the 3rd and the 4th digits respectively, that is they form the four-tuple 1975.

194. The answer to the first question posed in the problem can be found quite simply. It is clear that among the 8-digit numbers and the numbers consisting of a smaller number of digits there is no number the sum of whose digits is equal to $9 \cdot 9 = 81$ and that among the 9-digit numbers there is exactly one such number, namely 999 999 999; accordingly, in sequence (**) the number 81 occurs for the first time in the 111 111 111th place; the number following 81 in sequence (**) is equal to 9 (9 is equal to the sum of the digits of the number 1 000 000 008).

The answer to the second question can also be found in a rather simple way. Since among the numbers belonging to sequence (**) which correspond to the numbers in sequence (*) consisting of not more than three digits the number 27 occurs exactly once (in the 111th place in sequence (**); in sequence (*) this place is occupied by the number 999), it is clear that the number 27 can not repeat four times here. As to the 4-digit numbers of sequence (*), among them there are 4-tuples of consecutive numbers whose sums of digits are equal to 27: these are the numbers 3969; 3978; 3987 and 3996; it is evident that the corresponding numbers 27, 27, 27, 27 in sequence (**) precede the first triple of the numbers 36 in this sequence (they even precede the first number 36 corresponding to the number 9999 in sequence (*)).

The last question of the problem is stated rather ambiguously; however it is closely related to the way in which the four-tuple of the numbers 3969; 3978; 3987; 3996 indicated above can be determined. We do not go into detail here and only limit ourselves to indicating that the structure of the numbers forming sequence (**) is connected with the monotonicity of the sequence of digits in the numbers resulting from the division by 9 of the numbers belonging to sequence (*). In particular, the numbers $441 = 3969/9$; 442; 443; 444 corresponding to the four-tuple of numbers of sequence (*) considered above are such that their digits do not form increasing sequences. Let us write down all numbers from 1 to 110 as the following table in which the two-digit numbers whose digits (different from 0) *do not increase* and the three-digit numbers whose digits (different from 0) *do not decrease* are printed in bold face type:

1	2	3	4	5	6	7	8	9	10	**11**
12	13	14	15	16	17	18	19	20	**21**	**22**
23	24	25	26	27	28	29	30	**31**	**32**	**33**
34	35	36	37	38	39	40	**41**	**42**	**43**	**44**
45	46	47	48	49	50	**51**	**52**	**53**	**54**	**55**
56	57	58	59	60	**61**	**62**	**63**	**64**	**65**	**66**
67	68	69	70	**71**	**72**	**73**	**74**	**75**	**76**	**77**
78	79	80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**
89	90	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**
100	**101**	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**

On multiplying all these numbers by 9 and computing the sum of the digits for each of them we arrive at the following remarkable table:

9	9	9	9	9	9	9	9	9	9	**18**
9	9	9	9	9	9	9	9	9	**18**	**18**
9	9	9	9	9	9	9	9	**18**	**18**	**18**
9	9	9	9	9	9	9	**18**	**18**	**18**	**18**
9	9	9	9	9	9	**18**	**18**	**18**	**18**	**18**
9	9	9	9	9	**18**	**18**	**18**	**18**	**18**	**18**
9	9	9	9	**18**	**18**	**18**	**18**	**18**	**18**	**18**
9	9	9	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**
9	9	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**
9	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**

There also exist some other interesting configurations of the numbers forming sequence (**).

195. First of all we note that each of the collections I_0, I_1, I_2, ... is obtained from the preceding collection by adding to it several new numbers, all the numbers contained in the preceding collection entering into the new one. Further, it is readily seen that the new numbers appearing when we pass from I_{n-1} to the collection I_n are greater than n. Therefore the number 1973 does not occur in the collections with indices exceeding 1973, that is all such collections contain one and the same number of the numbers 1973. Now let us prove that *a fixed pair of numbers a and b* (where, for definiteness, a stands to the left of b; here two pairs of the form a, b and b, a are considered different) *occurs in the sequence I_0, I_1, I_2, ..., I_n ... of the collections exactly once in case the numbers a and b are relatively prime and does not occur at all in case a and b are not relatively prime.* This assertion is quite obvious for the pairs of numbers a, b the greatest of which does not exceed 2 (there are only two such pairs: 1, 2 and 2, 1; each of these pairs occurs only once, namely in the collection I_1 which consists of the numbers 1, 2, 1). We shall prove this assertion using the *method of mathematical induction*. Let us assume that the assertion has already been proved for all pairs of numbers a, b such that $\max [a, b] < n$ * and then show that under this assumption the assertion is also true for the pairs a, b such that $\max [a, b] = n$. Indeed, let a, b be a pair of positive integers, say such that $\max [a, b] = b = n$. It is clear that the pair a, b can appear in a collection I_k only if the preceding collection I_{k-1} contains a pair of numbers a and $b - a$ standing side by side. Now, since $\max [a, b - a] < \max [a, b] = b = n$, the above assumption implies that the pair of numbers a, $b - a$ occurs exactly once in the collections $I_1, \ldots, I_{k-1}$ when the numbers a and $b - a$ are relatively prime and does not occur in these collections when a and $b - a$ have a common divisor $d > 1$. It follows immediately that the pair a, b also occurs in the collections we are considering exactly once when the numbers a and b are relatively prime and does not occur at all when a and b are not relatively prime because *the numbers a and b are relatively prime if and only if so are the numbers a and $b - a$.*

Now it becomes clear that since 1973 is a prime number (let the reader check this), each of the pairs of numbers 1, 1972; 2, 1971; 3, 1970; ...; 1971, 2; 1972, 1 occurs exactly once in the collections under consideration because all these pairs consist of relatively prime numbers. It readily follows that the number 1973 is contained in the collections I_n with indices $n > 1973$ (and, in particular, in the collection $I_{1\,000\,000}$) exactly 1972 times (1972 is equal to the number of the pairs 1, 1972; 2, 1971; ...; 1972, 1).

Remark. From the solution of this problem it also follows that *an arbitrary natural number N occurs in the collections I_n* (where $n > N$) *exactly $\varphi(N)$ times where $\varphi(N)$ is the number of positive integers which are less than N and are relatively prime to N*; on the computation of the number $\varphi(N)$ (for any given N) see the remark to the condition of Problem 341.

196. Let $\alpha_1\alpha_2\alpha_3\alpha_4$ (where each α_i is equal to the digit 0 or 1; $i = 1, 2, 3, 4$) be the last four digits of the given sequence. If the sequence did not contain a subsequence of the digits $\alpha_1\alpha_2\alpha_3\alpha_4 0$ standing side by side it would be possible to continue the sequence by writing an additional digit 0 at its end. Similarly, if the sequence did not contain the five-tuple of the digits $\alpha_1\alpha_2\alpha_3\alpha_4 1$ it would be possible to write an additional digit 1 at the end. Therefore, the four-tuples of the consecutive digits $\alpha_1\alpha_2\alpha_3\alpha_4$ occur three

* The symbol $\max [a, b]$ designates the *greatest* of the numbers a and b in case $a \neq b$ and any of the numbers a and b in case $a = b$.

times in the given sequence (one of these four-tuples is followed by the digit 0, another four-tuple $\alpha_1\alpha_2\alpha_3\alpha_4$ is followed by the digit 1 and one more four-tuple $\alpha_1\alpha_2\alpha_3\alpha_4$ stands at the end of the given sequence). Two of these four-tuples are preceded by the digits 0 and 1 whereas the third four-tuple $\alpha_1\alpha_2\alpha_3\alpha_4$ cannot be preceded by *any* digit α_0 (because, if otherwise, the five-tuple of the digits $\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4$ would occur twice in the sequence). Therefore the third four-tuple $\alpha_1\alpha_2\alpha_3\alpha_4$ must stand *at the beginning* of the sequence.

197. The number N is *the product of all prime numbers from* 2 *to* 37 *inclusive*; every divisor of the number N is a product of some of these prime numbers. Let us show that the assertion of the problem is true for any number $N_k = 2\cdot3\cdot5\cdot7\cdot\ldots\cdot p_k$ equal to the product of the first k prime numbers *. We shall prove what has been said with the aid of the *method of mathematical induction* (with respect to the number k). It is clear that for $k = 1$ we have the number $N_1 = 2$ possessing only two divisors 1 and 2 below which the numbers $+1$ and -1 are written respectively; in this case we have $+1 + (-1) = 0$. Further, let us assume that the assertion has already been proved for the number N_k equal to the product of the first k prime numbers; in other words, we suppose that it has already been proved that the number N_k has an even number $2n$ of divisors (including the numbers 1 and N_k) among which there are n divisors each of which is a product of an even number of prime factors (let us agree to call these divisors "even"; in particular, the number 1 is a divisor of this kind since it has *zero* prime factors since 1 is neither a prime nor a composite natural number, the number zero being even) whereas each of the other n divisors is a product of an odd number of prime factors (we shall conditionally call them "odd" divisors of N_k). We shall prove that under this assumption the assertion of the problem is true for the number $N_{k+1} = N_k\cdot p_{k+1}$ as well where p_{k+1} is the $(k+1)$th prime number. It is evident that each of the divisors of the number N_k is also a divisor of the number N_{k+1}, and hence N_{k+1} has $2n$ divisors (which do not exhaust the set of all divisors of N_{k+1}) among which there are n "even" divisors and n "odd" divisors. In addition to these $2n$ divisors the number N_{k+1} has a number of divisors which are not divisors of the number N_k: these are the divisors of N_{k+1} which are divisible by p_{k+1}. They all can be obtained by multiplying all divisors of the number N_k by p_{k+1}. The n "even" divisors of N_k thus generate n "new" divisors of N_{k+1} each of which is a product of an odd number of prime factors and the n "odd" divisors of N_k generate n "even" divisors of N_{k+1}. Thus, the total number of the divisors

* This assertion even holds for all natural numbers which are factored as *products of pairwise distinct prime numbers.*

of the number N_{k+1} is equal to $4n$; among them there are $2n = n + n$ "even" divisors and $2n = n + n$ "odd" divisors. Therefore for N_{k+1} the numbers written in the lower line are $2n$ numbers $+1$'s and $2n$ numbers -1's; the sum of these numbers equal to $+1$ or to -1 is equal to 0.

Remark. The argument used in this solution allows us to state the assertion of the problem in a more precise manner. Namely, this argument shows that the number N_k has 2^{k-1} "even" and 2^{k-1} "odd" divisors (in particular, the number $N = N_{12}$ indicated in the condition of the problem has $2^{11} = 2048$ "even" and 2048 "odd" divisors) whence it follows that for N_k the sequence of numbers written in the lower line consists of 2^{k-1} numbers $+1$ (for the number N we have $2^{k-1} = 2^{11}$) and of 2^{k-1} numbers -1.

198. The fact that the numbers p and q are relatively prime makes it possible to use the Euclidean algorithm to prove that *any integer n can be represented in the form*

$$n = px + qy \tag{*}$$

where x and y are integers. Indeed, let $p > q$; then $p = qd + r$ where $0 < r < q$, that is

$$r = p \cdot 1 + q(-d) = px_1 + qy_1$$

where d is the quotient and r is the remainder resulting from the division of p by q, $x_1 = 1$ and $y_2 = -d$. Further, we have $q = rd_1 + r_1$ where $0 < r_1 < r$ (the number is equal to the remainder resulting from the division of q by r). On combining these two equalities we represent r_1 in form (*):

$$r_1 = q - rd_1 = q - (px_1 + qy_1)\,d_1 =$$
$$= p(-x_1d_1) + q(1 - y_1d_1) = px_2 + qy_2$$

where $x_2 = -x_1d_1 = -d_1$ and $y_2 = 1 - y_1d_1 = 1 + dd_1$ are integers. Next we put $r = r_1d_2 + r_2$ where $0 < r_2 < r_1$ and use the foregoing equality in order to represent in just the same way the number r_2 as a combination of the form $px_3 + qy_3$ of the numbers p and q, and so on until we arrive at the *greatest common divisor* 1 of the numbers p and q. (It is readily seen that in this process the last remainder different from 0 is the *greatest common divisor* for any two initial numbers p and q; in the case under consideration p and q are relatively prime and this greatest common divisor is equal to 1.) Now, since the number 1 can be written in the form $1 = px_k + qy_k$ where x_k and y_k are integers it follows that *any* number $n = n \cdot 1 = n(px_k + qy_k) = p(nx_k) + q(ny_k)$ can also be written in form (*).

Further, the fact that p and q are *relatively prime* implies that *if n can be represented in form* (*) *in two different ways, say*

$$n = px + qy = px' + qy' \tag{**}$$

where x, y, x' and y' are integers, then *the difference* $x - x'$ *is multiple of* q *and the difference* $y - y'$ *is multiple of* p. Indeed, relation (**) implies that $p(x-x') = -q(y-y')$, that is $x - x' = -q(y-y')/p$, and since p and q have no common prime factors the difference $x - x'$ is divisible by q. The last assertion means that representation (*) is "unique" in the sense that *every integer* n *can be uniquely represented in form* (*) *where* $0 \leqslant x < q$. Indeed, the number x in formula (*) can always be written in the form $x = kq + x_0$ where $0 \leqslant x_0 < q$ (the number x_0 is equal to the remainder resulting from the division of x by q), and we have

$$n = px + qy = p(kq + x_0) + qy = px_0 + q(pk + y) = px_0 + qy_0 \quad (***)$$

where $0 \leqslant x_0 < q$ and the numbers x_0 and y_0 are integers. On the other hand, *two different* representations (***), that is two equalities of form (**) with $0 \leqslant x < q$, $0 \leqslant x' < q$ and $x \neq x'$, do not exist because if they existed then we should have $|x - x'| < q$ and the difference $x - x'$ would not be divisible by q. The number n is obviously "good" when the number y_0 in formula (***) is nonnegative and is "bad" when y_0 is negative because if $n = px_0 + qy_0$ where $0 \leqslant x_0 < q$ and $y_0 < 0$ then the replacement of x_0 and y_0 by $x = x_0 - \lambda q$ and $y = x_0 + \lambda p$ respectively where λ is an integer can never lead to representation (*) of the number n in which both x and y are nonnegative.

What has been said makes it possible to easily solve the problem.

(a) It is evident that the smallest "good" number is the number $0 = 0 \cdot p + 0 \cdot q$; by virtue of the above, the greatest "bad" number is a number of form (***) where x_0 is the greatest of the "admissible" (positive) numbers, that is $x_0 = q - 1$, and y_0 is the smallest of the negative numbers, that is $y_0 = -1$. Thus, the greatest "bad" number is $P = p(q-1) + q(-1) = pq - p - q$. Further, it is natural to assume that these two numbers 0 and P are those whose sum is equal to the number A mentioned in the condition of the problem, that is it is natural to put A equal to $P = pq - p - q$. Indeed, if n is a number of form (***) then the number

$$n' = A - n = (pq - p - q) - (px_0 + qy_0) = \\ = p(q - 1 - x_0) + q(-1 - y_0)$$

is also of form (***) where the role of x_0 is played by the number $x_0' = q - 1 - x_0$ and the role of y_0 by the number $y_0' = -1 - y_0$. The relation $0 \leqslant x_0 \leqslant q - 1$ implies $0 \leqslant x_0' \leqslant q - 1$, and one of the numbers y_0 and y_0' is positive whereas the other is negative; this proves the assertion of the problem.

(b) *First solution.* From the result established in the solution of Problem 198 (a) it readily follows that exactly one half of the numbers n satisfying the inequality $0 \leqslant n \leqslant P = pq - p - q$ (where P is the greatest "bad" number) are "bad" while the others are "good". Since we thus exhaust all "bad" natural numbers, the number t of such numbers is given by the formula

$$t = \frac{P+1}{2} = \frac{pq-p-q+1}{2} = \frac{(p-1)(q-1)}{2}$$

Second solution. The "bad" numbers are those natural numbers n which can be represented in form (***) where $0 \leqslant x_0 < q$, $y_0 < 0$ and $px_0 + qy_0 > 0$ (because n is a positive number). Let us consider the plane with coordinates x_0 and y_0 shown in Fig. 22. The straight lines $x_0 = q$, $y_0 = 0$ and $px_0 + qy_0 = 0$ cut the triangle OAB from the plane (it is shaded in the figure). The problem reduces to the determination of the number of the points in the plane which have integral coordinates and lie within this triangle. The number we are interested in is clearly half that of the points with integral coordinates lying within the rectangle $OABC$ (there are no points with integral coordinates on the diagonal of the rectangle because p and q are relatively prime). Since the number of the points with integral coordinates lying within the rectangle $OABC$ is obviously equal to $(p-1)(q-1)$, we arrive at the same value of the sought-for number t as above.

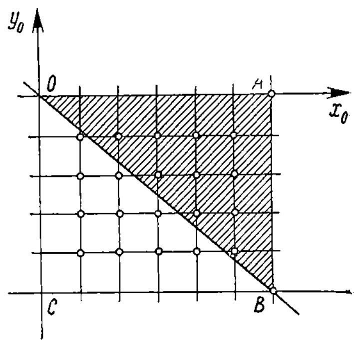


Fig. 22

199. *First solution.* We have to prove that every nonnegative integer n can be uniquely represented in the form

$$n = \frac{(x+y)^2 + 3x + y}{2} = \frac{(x+y)^2 + (x+y) + 2x}{2} =$$

$$= \frac{(x+y)^2 + (x+y)}{2} + x = \frac{X(X+1)}{2} + x$$

where $X = x + y$ and, consequently, since x and y are nonnegative numbers, $X \geqslant 0$ and $0 \leqslant x \leqslant X$. It is clear that for a fixed $X \geqslant 0$ and x varying within the admissible limit from 0 to X the number $n = X(X+1)/2 + x$ assumes all integral values from $N_X = X(X+1)/2$ (in this case $x = 0$) to $N_X = X(X+1)/2 + X$ (in this case $x = X$), each of these values occurring exactly once. The next integral number $N_X + 1 = X(X+1)/2 + X + 1 =$

$= (X+1)(X+2)/2$ corresponds to the value of X exceeding the former value by 1 (that is to the value $X+1$) and to the value $x=0$; in this case when x varies from 0 to $X+1$ we again obtain in just the same way all the integral numbers from $N_{X+1} = (X+1)(X+2)/2$ to $N'_{X+1} = N_{X+2} - 1$ etc. This argument proves the assertion stated in the problem.

Second solution. Let us index all the points in the plane having nonnegative integral coordinates (x, y) in the way indicated in Fig. 23. Let us prove that the point with coordinates x, y receives the index $n = [(x+y)^2 + 3x + y]/2$; this auxiliary assertion will imply the assertion of the problem because in the infinite sequence of the points (x, y) with nonnegative integral coordinates the point indexed by the number n will occur exactly once.

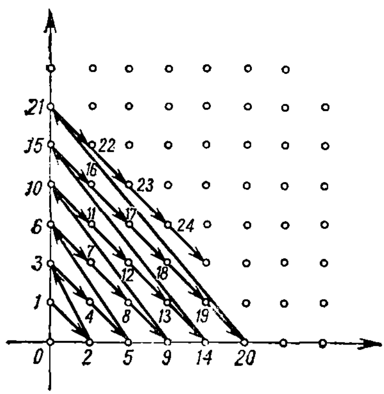


Fig. 23

We shall prove the auxiliary assertion *by induction* (with respect to n). It is evident that the index $n=0$ is assigned to the point $(0, 0)$ with zero coordinates and the index 1 to the point with coordinates $(0, 1)$; for the coordinates of these points we have

$$0 = \frac{(0+0)^2 + 3\cdot 0 + 0}{2}$$

and

$$1 = \frac{(0+1)^2 + 3\cdot 0 + 1}{2}$$

Now let us suppose that the auxiliary assertion has already been proved for all points belonging to the sequence whose indices range from 1 to n; let the nth point have coordinates x, y. If the coordinate y of this point is different from zero then the coordinates of the $(n+1)$th point are equal to $x+1$ and $y-1$, and for this point we do in fact have

$$\frac{[(x+1)+(y-1)]^2 + 3(x+1) + (y-1)}{2} = \frac{(x+y)^2 + 3x + y}{2} + 1 = n+1$$

If $y=0$, the coordinates of the $(n+1)$th point are $(0, x+1)$ and in this case we also have

$$\frac{(0+x+1)^2 + 3\cdot 0 + (x+1)}{2} = \frac{(x+0)^2 + 3x + 0}{2} + 1 = n+1$$

200. To every irreducible fraction p/q where

$$0 < p \leqslant 100 \quad \text{and} \quad 0 < q \leqslant 100 \qquad (*)$$

we shall assign a point M whose coordinates are p, q; inequalities (*) indicate that the point M belongs to the square $\mathcal{K} = OACB$ shaded in Fig. 24 which is bounded by the coordinate axes and by the straight lines $x = 100$ and $y = 100$ (the point M lies inside the square $\mathcal{K}$ or belongs to one of the sides AC and BC of that square). The irreducibility of the fraction p/q implies that the line segment OM contains no points with integral coordinates other than M (indeed, the equalities $p = kp_1$ and $q = kq_1$ showing that the fraction p/q can be reduced by a factor k mean that the point $M_1(p_1, q_1)$ with integral coordinates also belongs to the line segment OM). Further, if the line segment OP which lies on a straight line l passing through the origin O (P is the point of intersection of l with the side AC or with the side BC of the square $\mathcal{K}$) contains n points (p_0, q_0), $(2p_0, 2q_0)$, $(3p_0, 3q_0)$, ..., (np_0, nq_0) with integral coordinates (among them $M_0(p_0, q_0)$ is the point lying at the shortest distance from 0) then (since $np_0 \leqslant 100$ and $nq_0 \leqslant 100$) we have

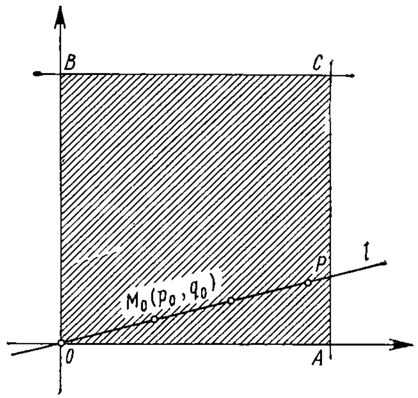


Fig. 24

$$p_0 \leqslant \frac{100}{n} < \frac{100}{n-1} < \frac{100}{n-2} < \ldots < \frac{100}{1}$$

and

$$q_0 \leqslant \frac{100}{n} < \frac{100}{n-1} < \frac{100}{n-2} < \ldots < \frac{100}{1}$$

This means that the irreducible fraction p_0/q_0 is taken into account when the terms $d\left(\frac{100}{1}\right)$, $d\left(\frac{100}{2}\right)$, ..., $d\left(\frac{100}{n}\right)$ of the sum S are computed, that is the total number of times the fraction p_0/q_0 is taken into account when the sum S is computed equals n (whereas the reducible fractions $2p_0/2q_0$, $3p_0/3q_0$, ..., np_0/nq_0 are not of course taken into account in the computation of the sum S). Thus, *the input which the fraction p_0/q_0 gives to the sum S is equal to n*, that *is to the number of the points with integral coordinates which belong to the line segment OP.*

What has been proved implies that the total number of the fractions which are taken into account when the sum S is computed (this number is equal to the sum S) is equal to the total number of all points with integral coordinates lying within the square $\mathcal{K}$, that is $S = 100 \cdot 100 = 10\,000$.

201. (1) Let x be an arbitrary real number. Then we can write $x=[x]+\alpha$ where α is a nonnegative number less than 1. Now let us represent y in the form $y=[y]+\beta$ $(0 \leqslant \beta < 1)$. Then $x+y=[x]+[y]+\alpha+\beta$. Since $\alpha+\beta \geqslant 0$ the last equality shows that $[x]+[y]$ is an integer not exceeding $x+y$. Further, since $(x+y)$ is the *greatest* of the integers not exceeding $x+y$ we have $[x+y] \geqslant [x]+[y]$.

(2) *First solution.* Let us represent x in the form $x=[x]+\alpha$ where $0 \leqslant \alpha < 1$. The division of the integer $[x]$ by n results in a quotient q and a remainder r, that is $[x]=qn+r (0 \leqslant r \leqslant n-1)$. Thus, we have

$$\frac{[x]}{n}=q+\frac{r}{n}, \quad \left[\frac{[x]}{n}\right]=q \quad \text{and} \quad x=qn+r+\alpha=qn+r_1$$

where $r_1=r+\alpha < n$. Hence, $x/n=q+r_1/n$ $(0 \leqslant r_1/n < 1)$ and $[x/n]=q=[[x]/n]$, which is what we intended to prove.

Second solution. Let us consider all whole numbers which do not exceed x and are divisible by n. The number of these whole numbers is obviously equal to $[x/n]$. Let us also consider all whole numbers which do not exceed $[x]$ and are divisible by n. Their number is equal to $[[x]/n]$. Now, since these groups of whole numbers coincide, the numbers of the members in these groups are equal, and consequently $[[x]/n]=[x/n]$.

(3). If $(x)=[x]$ (that is $x-[x] < 1/2$) then $[x+1/2]=[x]$, and we have $[2x]=2[x]$ and $[2x]-[x]=2[x]-[x]=[x]=[x+1/2]$. If $(x)=[x]+1$ (that is $x-[x] \geqslant 1/2$) then $[x+1/2]=[x]+1$, $[2x]=2[x]+1$, and we again have $[2x]-[x]=2[x]+1-[x]=[x]+1=[x+1/2]$.

(4) *First solution.* Let us write x in the form $x=[x]+\alpha$. Since $0 \leqslant \alpha < 1$, the number α lies between two neighbouring fractions belonging to the set $0/n, 1/n, \ldots, (n-1)/n, n/n$. Let these neighbouring fractions be k/n and $(k+1)/n$, that is let $k/n \leqslant \alpha < (k+1)/n$; then we have

$$x+\frac{n-k-1}{n}=[x]+\alpha+\frac{n-k-1}{n}<$$
$$<[x]+\frac{k+1}{n}+\frac{n-k-1}{n}=[x]+1$$

and

$$x+\frac{n-k}{n}=[x]+\alpha+\frac{n-k}{n} \geqslant [x]+\frac{k}{n}+\frac{n-k}{n}=[x]+1$$

Further,

$$x+\frac{n-1}{n}=[x]+\alpha+\frac{n-1}{n}<[x]+\frac{k+1}{n}+\frac{n-1}{n}=$$
$$=[x]+\frac{n+k}{n}<[x]+2$$

It folows that

$$[x] \leqslant \left[x+\frac{1}{n}\right] \leqslant \left[x+\frac{2}{n}\right] \leqslant \ldots \leqslant \left[x+\frac{n-k-1}{n}\right] < [x]+1$$

and

$$[x]+1 \leqslant \left[x+\frac{n-k}{n}\right] \leqslant \left[x+\frac{n-k+1}{n}\right] \leqslant \ldots$$
$$\ldots \leqslant \left[x+\frac{n-1}{n}\right] < [x]+2$$

that is

$$[x] = \left[x+\frac{1}{n}\right] = \left[x+\frac{2}{n}\right] = \ldots = \left[x+\frac{n-k-1}{n}\right]$$

and

$$\left[x+\frac{n-k}{n}\right] = \left[x+\frac{n-k+1}{n}\right] = \ldots = \left[x+\frac{n-1}{n}\right] = [x]+1$$

Since the first group of the equalities involves $n-k$ numbers while the second group involves k numbers we have

$$[x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=$$
$$=(n-k)[x]+k([x]+1)=n[x]+k$$

The integral part of the number nx is equal to the same number $n[x]+k$. Indeed, since $k \leqslant n\alpha < k+1$, we have $n\alpha = k+\beta$ where $0 \leqslant \beta < 1$, and consequently

$$[nx]=[n[x]+n\alpha]=[n[x]+k+\beta]=n[x]+k$$

We have thus proved that

$$[x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$$

Second solution. Let us consider the left-hand side of the given equality. If $0 \leqslant x < 1/n$, all the numbers x, $x+1/n$, ..., $x+(n-1)/n$ are less than 1 and their integral parts are equal to 0. In this case $[nx]$ is also equal to 0, and consequently the equality holds for all x satisfying the condition $0 \leqslant x < 1/n$.

Now let x be an arbitrary number. If x is increased by $1/n$ then all the summands on the left-hand side shift one place to the right and the last summand $[x+(n-1)/n]$ turns into the number $[x+1]$ which exceeds $[x]$ by 1. Consequently, when x receives an increment of $1/n$ the left-hand side increases by 1. The increase of x by $1/n$ also results in an increment equal to 1 of the right-hand side of the equality. Further, for any x there is a number α lying between 0 and $1/n$ ($0 \leqslant \alpha < 1/n$) such that x differs from α by m/n where m is a whole number, whence it follows that the given equality is valid for any x.

202. *First solution.* By virtue of the result established in the solution of Problem 201 (3), the sum we are interested in (it obviously contains only a finite number of terms different from zero) is equal to

$$\left[\frac{n}{2}+\frac{1}{2}\right]+\left[\frac{n}{4}+\frac{1}{2}\right]+\left[\frac{n}{8}+\frac{1}{2}\right]+\ldots$$
$$=\left([n]-\left[\frac{n}{2}\right]\right)+\left(\left[\frac{n}{2}\right]-\left[\frac{n}{4}\right]\right)+\left(\left[\frac{n}{4}\right]-\left[\frac{n}{8}\right]\right)+\ldots=[n]=n$$

Second solution. For $n=1$ the sum under consideration is obviously equal to 1. Further, when n is replaced by $n+1$ every

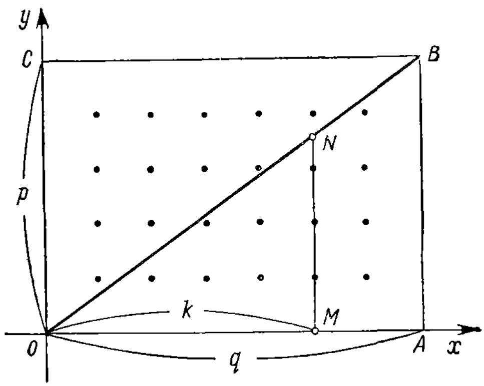


Fig. 25

term of the sum either remains unchanged or increases by 1. More precisely, there is *exactly one* term in this sum which increases by 1 under this operation, namely the term corresponding to the value of k such that 2^k is the highest power of 2 by which $n+1$ is divisible. Indeed, if $n+1=2^k(2m+1)$ then

$$\left[\frac{n+1+2^k}{2^{k+1}}\right]=\left[\frac{2^k(2m+2)}{2^{k+1}}\right]=m+1$$

and

$$\left[\frac{n+2^k}{2^{k+1}}\right]=\left[m+1-\frac{1}{2^{k+1}}\right]=m$$

whereas for $i\neq k$ we have $[(n+1+2^i)/2^{i+1}]=[(n+2^i/2^{i+1}]$ (why?). By the *principle of mathematical induction*, it follows that the sum under consideration is equal to n for all n.

203. Let us mark all the points in the plane xOy whose both coordinates are integers such that $1\leqslant x\leqslant q-1$ and $1\leqslant y\leqslant$ $\leqslant p-1$ (here x and y are the coordinates of the points). These points lie inside the rectangle $OABC$ (see Fig. 25) the lengths of whose sides are $OA = q$ and $OC = p$; the total number of these points is equal to $(q-1)(p-1)$. Let us draw the diagonal OB of the rectangle. It is clear that none of the points with integral coordinates lies on that diagonal (because the coordinates x and y of the points belonging to the diagonal OB are connected by the relation $x/y = OA/AB = q/p$, and, since q and p are relatively prime numbers, there are no positive integers $x < p$ and $y < q$ such that $x/y = q/p$). Now we note that the number of the points with integral coordinates whose abscissa is equal to k (where k is a positive integer smaller than q) and which lie below the diagonal OB is equal to the *integral part* of the length of the line segment MN shown in Fig. 25. Since $MN = (OM/OA)\cdot AB = kp/q$, this number is equal to $[kp/q]$. Consequently, the sum

$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+\left[\frac{3p}{q}\right]+\dots+\left[\frac{(q-1)p}{q}\right]$$

is equal to the *total* number of all points with integral coordinates lying below the diagonal OB. The total number of the points with integral coordinates lying inside the rectangle $OABC$ is equal to $(q-1)(p-1)$; the symmetry of the location of these points about the centre of the rectangle implies that exactly half of these points lies below the diagonal (in this argument it is important to take into account that the diagonal itself contains no points with integral coordinates). Thus,

$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+\left[\frac{3p}{q}\right]+\dots+\left[\frac{(q-1)p}{q}\right]=\frac{(q-1)(p-1)}{2}$$

In just the same way it is proved that

$$\left[\frac{q}{p}\right]+\left[\frac{2q}{p}\right]+\left[\frac{3q}{p}\right]+\dots+\left[\frac{(p-1)q}{p}\right]=\frac{(p-1)(q-1)}{2}$$

204. *First solution.* For $n = 1$ the right-hand side and the left-hand side of each of the given equalities reduces to exactly one term equal to 1; hence, for $n = 1$ these equalities hold. Now let us prove that if the given equalities hold for a given value of n then they also hold for $n + 1$; by virtue of the *principle of mathematical induction*, this will imply that the equalities hold for all values of n.

If $n + 1$ is not exactly divisible by k, that is if

$$n+1 = qk + r$$

where the remainder r lies between 1 and $k-1$, then $n = qk + r'$ where $r' = r - 1$, that is $0 \leqslant r' \leqslant k-2$. It follows that in this case the numbers $[(n+1)/k]$ and $[n/k]$ coincide (they are both equal to q). In case $n+1$ is divisible by k (that is $n + 1 = qk$) we obviously have $[(n+1)/k] = q$ and $[n/k] = q-1$, that is

$[(n+1)/k] = [n/k] + 1$. Thus,

$$\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right] \quad \text{if } k \text{ is not a divisor of the number } n+1$$

and

$$\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right] + 1 \quad \text{if } k \text{ is a divisor of the number } n+1.$$

Now it follows that:

(a) $$\left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \ldots + \left[\frac{n+1}{n+1}\right] = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \ldots + \left[\frac{n}{n+1}\right] + \tau_{n+1}$$

that is if

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \ldots + \left[\frac{n}{n}\right] = \tau_1 + \tau_2 + \ldots + \tau_n$$

then

$$\left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \ldots + \left[\frac{n+1}{n+1}\right] = \tau_1 + \tau_2 + \ldots + \tau_n + \tau_{n+1}$$

(b) $$\left[\frac{n+1}{1}\right] + 2\left[\frac{n+1}{2}\right] + \ldots + (n+1)\left[\frac{n+1}{n+1}\right] = \left[\frac{n}{1}\right] + 2\left[\frac{n}{2}\right] + \ldots + (n+1)\left[\frac{n}{n+1}\right] + \sigma_{n+1}$$

that is if

$$\left[\frac{n}{1}\right] + 2\left[\frac{n}{2}\right] + \ldots + n\left[\frac{n}{n}\right] = \sigma_1 + \sigma_2 + \ldots + \sigma_n$$

then

$$\left[\frac{n+1}{1}\right] + 2\left[\frac{n+1}{2}\right] + \ldots + (n+1)\left[\frac{n+1}{n+1}\right] = \sigma_1 + \sigma_2 + \ldots + \sigma_n + \sigma_{n+1}$$

Second solution. The number of those members of the sequence 1, 2, 3, ..., n which are divisible by a definite number k is equal to $[n/k]$ (these are the numbers k, $2k$, $3k$, ..., $[n/k]k$). The sum of the divisors equal to k of all such numbers is equal to $k[n/k]$. Now it follows that:

(a) The numerical value of the sum $[n/1] + [n/2] + \ldots + [n/k] + \ldots + [n/n]$ is equal to the number of those terms of the sequence 1, 2, 3, ..., n which are divisible by 1 plus the number of those terms of the sequence which are divisible by 2 ... plus the number of the terms of this sequence which are divisible by n, and the sum $\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n$ has the same numerical value.

(b) The numerical value of the sum $1 \cdot [n/1] + 2[n/2] + \ldots + k[n/k] + \ldots + n[n/n]$ is equal to the sum of the divisors equal to 1 of all the numbers belonging to the sequence 1, 2, 3, ..., n plus the sum of the divisors equal to 2 of all the numbers belonging to this sequence plus the sum of the divisors equal to 3 of the numbers belonging to that sequence ... plus the sum of the divisors equal to n of all the numbers of the sequence, and the sum $\sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_n$ has the same value.

Third solution. We shall also present here a simple geometrical solution of the problem whose basic idea is close to that of the

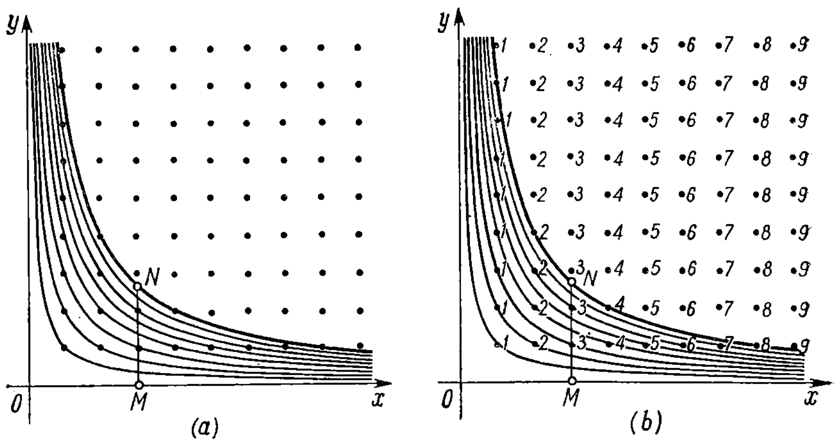


Fig. 26

second solution. Let us consider an equilateral hyperbola described by the equation $y = k/x$ (or, which is the same, by the equation $xy = k$; such a hyperbola serves as the graph representing the inverse proportionality). More precisely, we are interested in the part of the hyperbola lying in the first quadrant (see Fig. 26a).

Let us mark all the points with integral coordinates belonging to the first quadrant. If x is a divisor of the number k then there is a point with abscissa x on the equilateral hyperbola $xy = k$. Conversely, if a hyperbola described by an equation $xy = k$ passes through a point with integral coordinates whose abscissa is equal to x then x is a divisor of the number k. Thus, the number τ_k of the divisors of the number k is equal to the number of the points with integral coordinates lying on the hyperbola $xy = k$. The sum σ_k of the divisors of the number k is equal to the sum of the abscissas of the points with integral coordinates lying on the hyperbola $xy = k$. Further, we shall also use the fact that all the hyperbolas $xy = 1$, $xy = 2$, $xy = 3$, ..., $xy = n - 1$ lie below the hyperbola $xy = n$. Now we can draw the following conclusions.

(a) The sum $\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n$ is equal to the number of all points with integral coordinates lying below the hyperbola $xy = n$ and on the hyperbola itself. On the other hand, none of these points has an abscissa exceeding n; as to the number of the points with integral coordinates whose abscissas are equal to k and which lie below the hyperbola, it is equal to the integral part of the length of the line segment MN shown in Fig. 26a, that is this number is equal to $[n/k]$ because $MN = n/k$ (cf. the solution of Problem 203). Thus, we have

$$\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \ldots + \left[\frac{n}{n}\right]$$

(b) Let us assign to every point with integral coordinates an index equal to its abscissa (see Fig. 26b). Then the sum $\sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_n$ is equal to the sum of the indices of all points with integral coordinates lying below the hyperbola $xy = n$. On the other hand, the sum of the indices of all such points whose abscissas are k is equal to $k[n/k]$. Thus, we have

$$\sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_n = \left[\frac{n}{1}\right] + 2\left[\frac{n}{2}\right] + 3\left[\frac{n}{3}\right] + \ldots + n\left[\frac{n}{n}\right]$$

205. The expression $(2+\sqrt{2})^n + (2-\sqrt{2})^n$ is obviously equal to an integral number, for, if $(2+\sqrt{2})^n = a_n + b_n\sqrt{2}$ where a_n and b_n are whole numbers, then $(2-\sqrt{2})^n = a_n - b_n\sqrt{2}$ (this follows from Newton's binomial formula and can also be proved by means of the method of mathematical induction). Since $(2-\sqrt{2})^n < 1$, it follows that

$$[(2+\sqrt{2})^n] = (2+\sqrt{2})^n + (2-\sqrt{2})^n - 1$$

and, consequently,

$$(2+\sqrt{2})^n - [(2+\sqrt{2})^n] = 1 - (2-\sqrt{2})^n$$

Since $(2-\sqrt{2}) < 1$, the expression $(2-\sqrt{2})^n$ can be made arbitrarily small by taking a sufficiently large exponent n. If we chose n such that $(2-\sqrt{2})^n < 0.000001$ then

$$(2+\sqrt{2})^n - [(2+\sqrt{2})^n] = 1 - (2-\sqrt{2})^n > 0.999999$$

206. (a) *First solution.* Since $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is a whole number and since $(2-\sqrt{3})^n < 1$, we have

$$[(2+\sqrt{3})^n] = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 1$$

(cf. the solution of Problem 205). Using Newton's binomial formula we can open the parentheses in the expression $(2+\sqrt{3})^n+(2-\sqrt{3})^n$ to obtain

$$(2+\sqrt{3})^n+(2-\sqrt{3})^n=$$
$$=2\left(2^n+C(n,\ 2)\,2^{n-2}\cdot 3+C(n,\ 4)\,2^{n-4}\cdot 3^2+\ldots\right)$$

It follows that this expression is divisible by 2, and consequently the number $[(2+\sqrt{3})^n]=(2+\sqrt{3})^n+(2-\sqrt{3})^n-1$ is odd.

Second solution. The number $(2+\sqrt{3})^n$ can be represented in the form $a_n+b_n\sqrt{3}$ where a_n and b_n are whole numbers. Let us prove that

$$a_n^2-3b_n^2=1$$

To this end we shall apply the method of mathematical induction.

First of all, for $n=1$ we have $a_1=2$, $b_1=1$ and $2^2-3\cdot 1=1$. Further, let us suppose that

$$(2+\sqrt{3})^n=a_n+b_n\sqrt{3}$$

for some n where $a_n^2-3b_n^2=1$. Then we can write

$$(2+\sqrt{3})^{n+1}=(a_n+b_n\sqrt{3})(2+\sqrt{3})=$$
$$=(2a_n+3b_n)+(a_n+2b_n)\sqrt{3}$$

whence $a_{n+1}=2a_n+3b_n$ and $b_{n+1}=a_n+2b_n$. Consequently,

$$a_{n+1}^2-3b_{n+1}^2=(2a_n+3b_n)^2-3(a_n+2b_n)^2=a_n^2-3b_n^2=1$$

We have thus proved that $a_n^2-3b_n^2=1$ for any n.

It follows that

$$[a_n+b_n\sqrt{3}]=a_n+[b_n\sqrt{3}]=a_n+[\sqrt{3b_n^2}]=$$
$$=a_n+[\sqrt{a_n^2-1}]=a_n+(a_n-1)=2a_n-1$$

which means that the number $[(2+\sqrt{3})^n]=[a_n+b_n\sqrt{3}]$ is odd.

(b) Let us check that

$$[(1+\sqrt{3})^n]=\begin{cases}(1+\sqrt{3})^n+(1-\sqrt{3})^n-1 & \text{for even } n\\ (1+\sqrt{3})^n+(1-\sqrt{3})^n & \text{for odd } n\end{cases}$$

Indeed, the sums on the right-hand side of this formula are whole numbers (cf. the first solution of Problem 206 (a)). For an even n we have $0<(1-\sqrt{3})^n<1$ and for an odd n we have $-1<(1-\sqrt{3})^n<0$.

Now let us consider separately the cases when n is even and n is odd.

(1) Let n be even: $n = 2m$; then

$$[(1+\sqrt{3})^{2m}] = (1+\sqrt{3})^{2m} + (1-\sqrt{3})^{2m} - 1 =$$
$$= \{(1+\sqrt{3})^2\}^m + \{(1-\sqrt{3})^2\}^m - 1 =$$
$$= (4+2\sqrt{3})^m + (4-2\sqrt{3})^m - 1 =$$
$$= 2^m\{(2+\sqrt{3})^m + (2-\sqrt{3})^m\} - 1$$

The expression in the curly brackets is obviously equal to an integral number and consequently the number $[(1+\sqrt{3})^{2m}] = 2^m N - 1$ is always odd. Hence, when n is even the highest exponent of the power of 2 by which $[(1+\sqrt{3})^n]$ is divisible is equal to zero.

(2) Let n be odd: $n = 2m + 1$; then

$$[(1+\sqrt{3})^{2m+1}] = (1+\sqrt{3})^{2m+1} + (1-\sqrt{3})^{2m+1} =$$
$$= (4+2\sqrt{3})^m(1+\sqrt{3}) + (4-2\sqrt{3})^m(1-\sqrt{3}) =$$
$$= 2^m\{(2+\sqrt{3})^m(1+\sqrt{3}) + (2-\sqrt{3})^m(1-\sqrt{3})\} =$$
$$= 2^m\{((2+\sqrt{3})^m + (2-\sqrt{3})^m) + \sqrt{3}((2+\sqrt{3})^m - (2-\sqrt{3})^m)\}$$

Let $(2+\sqrt{3})^m = a_m + b_m\sqrt{3}$ where a_m and b_m are integers; then $(2-\sqrt{3})^m = a_m - b_m\sqrt{3}$. On substituting these expressions into the above formula we find

$$[(1+\sqrt{3})^{2m+1}] = 2^m\{a_m + b_m\sqrt{3} + a_m - b_m\sqrt{3} +$$
$$+ \sqrt{3}(a_m + b_m\sqrt{3} - a_m + b_m\sqrt{3})\} =$$
$$= 2^m(2a_m + 6b_m) = 2^{m+1}(a_m + 3b_m)$$

Now let us show that the number $a_m + 3b_m$ is odd. Indeed, we have

$$(a_m + 3b_m)(a_m - 3b_m) = a_m^2 - 9b_m^2 = (a_m^2 - 3b_m^2) - 6b_m^2 = 1 - 6b_m^2$$

(see the second solution of Problem 206 (a)). Since the number $1 - 6b_m^2$ is odd such are both factors $(a_m + 3b_m)$ and $(a_m - 3b_m)$. Consequently, the exponent of the highest power of 2 by which $[(1+\sqrt{3})^n]$ is divisible for an odd $n = 2m + 1$ is equal to

$$m + 1 = \frac{n+1}{2} = \left[\frac{n}{2}\right] + 1$$

207. The fact that the expression $(2+\sqrt{5})^p+(2-\sqrt{5})^p$ is an integral number and the inequalities $-1<(2-\sqrt{5})^p<0$ (they hold because p is odd) imply that

$$[(2+\sqrt{5})^p]=(2+\sqrt{5})^p+(2-\sqrt{5})^p$$

(cf. the solutions of Problems 205 and 206). By Newton's binomial formula,

$$(2+\sqrt{5})^p+(2-\sqrt{5})^p=$$
$$=2\left(2^p+C(p,2)\,2^{p-2}5+C(p,4)\,2^{p-4}5^2+\ldots+C(p,p-1)\,2\cdot 5^{\frac{p-1}{2}}\right)$$

and therefore

$$[(2+\sqrt{5})^p]-2^{p+1}=$$
$$=2\left(C(p,2)\,2^{p-2}5+C(p,4)\,2^{p-4}5^2+\ldots+C(p,p-1)\,2\cdot 5^{\frac{p-1}{2}}\right)$$

All the binomial coefficients

$$C(p,2)=\frac{p(p-1)}{1\cdot 2},\quad C(p,4)=\frac{p(p-1)(p-2)(p-3)}{1\cdot 2\cdot 3\cdot 4},\ \ldots$$
$$\ldots,\ C(p,p-1)=p$$

are divisible by the prime number p because the numerator in the expression for $C(p,k)$ is divisible by p whereas the denominator is not. Consequently, the difference $[(2+\sqrt{5})^p]-2^{p+1}$ is also divisible by p, which is what we had to prove.

208. We have

$$C(n,p)=\frac{n(n-1)(n-2)\ldots(n-p+1)}{p!}$$

Among the p consecutive whole numbers $n, n-1, n-2, \ldots, n-p+1$ there is only one number divisible by p; let us denote it by the letter N. Then we can write $[n/p]=N/p$, and the difference mentioned in the condition of the problem assumes the form

$$\frac{n(n-1)\ldots(N+1)N(N-1)\ldots(n-p+1)}{p!}-\frac{N}{p}$$

Now we note that the division of the numbers $n, n-1, \ldots, N+1, N-1, \ldots, n-p+1$ by p leaves all the possible remainders $1, 2, 3, \ldots, p-1$ (when the p consecutive whole numbers from $n-p+1$ to n are divided by p we obtain *all* the remainders $0, 1, 2, \ldots, p-1$, each of them occurring exactly once). It follows that the difference

$$n(n-1)\ldots(N+1)(N-1)\ldots(n-p+1)-(p-1)!$$

is divisible by p (to prove this it is sufficient to perform the term-by-term multiplication of all the equalities $n = k_1 p + a_1$, $n - 1 = k_2 p + a_2$, ..., $N + 1 = k_i p + a_i$, $N - 1 = k_{i+1} p + a_{i+1}$, ..., $n - p + 1 = k_{p-1} p + a_{p-1}$ where $k_1, k_2, \ldots, k_{p-1}$ are integers and the numbers $a_1, a_2, \ldots, a_{p-1}$ are equal to the numbers $1, 2, \ldots, p - 1$ taken in some unknown order). On multiplying this difference by the whole number N/p we obtain the new difference

$$\frac{n(n-1)\ldots(n-p+1)}{p} - \frac{N(p-1)!}{p}$$

which is of course also divisible by p. Finally, on dividing both members of the last difference by $(p-1)!$ we arrive at the required result (the quotient resulting from the division by $(p-1)!$ is also divisible by p because the numbers $(p-1)!$ and p are relatively prime).

209. Let $\alpha > 0$. It is clear that the value $\alpha = 1$ satisfies the condition of the problem. In the case when $\alpha > 1$ and, accordingly, $1/\alpha = \beta < 1$, the numbers $[\alpha]$, $[2\alpha]$, $[3\alpha]$, ..., $[N\alpha]$ are all pairwise different, and therefore it only remains to check that all the numbers $[\beta]$, $[2\beta]$, $[3\beta]$, ..., $[N\beta]$ are different from one another. Since $[N\beta] \leqslant N\beta < N$ we have $[N\beta] \leqslant N - 1$, and therefore the N nonnegative numbers $[\beta]$, $[2\beta]$, ..., $[N\beta]$ can assume N different values only when these values are

$$[\beta] = 0, \quad [2\beta] = 1, \quad]3\beta] = 2, \ldots, [N\beta] = N - 1 \qquad (*)$$

Further, since the equality $[k\beta] = k - 1$ is equivalent to the inequalities $k - 1 \leqslant k\beta < k$ or, which is the same, to the inequalities

$$1 - \frac{1}{k} \leqslant \beta < 1 \quad (k = 1, 2, \ldots, N) \qquad (**)$$

we conclude that the system of inequalities (*) is equivalent to inequalities (**), that is to the inequalities

$$1 - \frac{1}{N} \leqslant \beta < 1$$

Hence, if $\alpha > 1$ then $(N-1)/N \leqslant \beta = 1/\alpha < 1$, and consequently $1 < \alpha \leqslant N/(N-1)$, and if $\alpha < 1$ then $(N-1)/N \leqslant \alpha < 1$ (this was in fact already proved above).

Now, taking into account the value $\alpha = 1$ mentioned at the beginning of the solution of this problem we see that for $\alpha > 0$ there must be $(N-1)/N \leqslant \alpha \leqslant N/(N-1)$. It can be shown in a similar manner that for $\alpha < 0$ there must be $-(N-1)/N \geqslant \alpha \geqslant -N/(N-1)$.

210. *First solution.* It is evident that $(a) = [a + 1/2]$; consequently, the equality we have to prove can be written in the form

$$N = \left[\frac{N}{2} + \frac{1}{2}\right] + \left[\frac{N}{4} + \frac{1}{2}\right] + \left[\frac{N}{8} + \frac{1}{2}\right] + \ldots$$

Now, let

$$N = a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \ldots + a_1 \cdot 2 + a_0$$

(where each of the digits $a_n, a_{n-1}, \ldots, a_1, a_0$ is equal to 0 or 1) be the expansion of the number N in powers of 2 (this corresponds to the binary representation of the number N). It obviously follows that

$$\left[\frac{N}{2} + \frac{1}{2}\right] = \left[a_n \cdot 2^{n-1} + a_{n-1} \cdot 2^{n-2} + \ldots + a_1 + \frac{a_0 + 1}{2}\right] =$$
$$= a_n \cdot 2^{n-1} + a_{n-1} \cdot 2^{n-2} + \ldots + a_1 + a_0$$

$$\left[\frac{N}{4} + \frac{1}{2}\right] = \left[a_n \cdot 2^{n-2} + a_{n-1} \cdot 2^{n-3} + \ldots + \frac{a_1 + 1}{2} + \frac{a_0}{4}\right] =$$
$$= a_n \cdot 2^{n-2} + a_{n-1} \cdot 2^{n-3} + \ldots + a_1$$

. . . .

$$\left[\frac{N}{2^n} + \frac{1}{2}\right] = \left[a_n + \frac{a_{n-1} + 1}{2} + \frac{a_{n-2}}{4} + \ldots + \frac{a_0}{2^n}\right] = a_n + a_{n-1}$$

$$\left[\frac{N}{2^{n+1}} + \frac{1}{2}\right] = \left[\frac{a_0 + 1}{2} + \frac{a_{n-1}}{4} + \ldots + \frac{a_n}{2^{n+1}}\right] = a_n$$

and

$$\left[\frac{N}{2^{n+2}} + \frac{1}{2}\right] = \left[\frac{N}{2^{n+3}} + \frac{1}{2}\right] = \ldots = 0$$

(we remind the reader that each of the digits $a_n, \ldots, a_0$ is equal to 0 or 1). Thus, we obtain

$$\left[\frac{N}{2} + \frac{1}{2}\right] + \left[\frac{N}{4} + \frac{1}{2}\right] + \ldots + \left[\frac{N}{2^{n+1}} + \frac{1}{2}\right] + \ldots$$
$$\ldots = a_n\left(2^{n-1} + 2^{n-2} + \ldots + 1 + 1\right) +$$
$$+ a_{n-1}\left(2^{n-2} + 2^{n-3} + \ldots + 1 + 1\right) + \ldots + a_1(1 + 1) + a_0 =$$
$$= a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \ldots + a_1 \cdot 2 + a_0 = N$$

which is what we had to prove.

Second solution. It is evident that the number of those odd numbers which do not exceed N is equal to $N/2$ in case N is even and is equal to $(N+1)/2 = [N/2] + 1$ in case N is odd, that is, it is always equal to $(N/2)$. Similarly, the number of those even numbers which do not exceed N and are not divisible by 4 is equal to $[N/4]$ in case N is divisible by 4 or in case its division by 4 leaves a remainder of 1 and is equal to $[N/4] + 1$ in case the division of N by 4 leaves a remainder equal to 2 or 3; in other words, this number is always equal to $(N/4)$. Further, the number of those numbers which do not exceed N and are divisible by 4 and not divisible by 8 is equal to $[N/8]$ in case N is divisible by 8 or in case its division by 8 leaves a remainder equal to 1 or 2 or 3 and is equal to $[N/8]+1$ in all the other cases; hence, this number is always equal to $(N/8)$. In an analogous manner we can prove that $(N/16)$ is equal to the number of those numbers not exceeding N which are divisible by 8 and are not divisible by 16, $(N/32)$ is equal to the number of those numbers not exceeding N which are divisible by 16 and not divisible by 32 etc. In this way we obviously enumerate all whole numbers from 1 to N, and consequently

$$\left(\frac{N}{2}\right)+\left(\frac{N}{4}\right)+\left(\frac{N}{8}\right)+\ldots=N$$

which is what we had to prove.

211. Since $2^{10}=1024$, we have $2^{100}=1024^{10}$. The decimal representation of the number $1000^{10}=10^{30}$ consists of one digit 1 and 30 noughts, and $1024^{10}>1000^{10}$; therefore the number $2^{100}=1024^{10}$ cannot have less than 31 digits. On the other hand,

$$\frac{1024^{10}}{1000^{10}}<\left(\frac{1025}{1000}\right)^{10}=\left(\frac{41}{40}\right)^{10}=\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}\cdot\frac{41}{40}<$$

$$<\frac{41}{40}\cdot\frac{40}{39}\cdot\frac{39}{38}\cdot\frac{38}{37}\cdot\frac{37}{36}\cdot\frac{36}{35}\cdot\frac{35}{34}\cdot\frac{34}{33}\cdot\frac{33}{32}\cdot\frac{32}{31}=\frac{41}{31}<10$$

since

$$\frac{41}{40}<\frac{40}{39}<\frac{39}{38}<\ldots<\frac{33}{32}<\frac{32}{31}$$

$\left(\text{because } \frac{41}{40}=1+\frac{1}{40},\ \frac{40}{39}=1+\frac{1}{39},\ \ldots,\ \frac{33}{32}=1+\frac{1}{32},\ \frac{32}{31}=\right.$ $\left.=1+\frac{1}{31}\right)$.

Thus,

$$2^{100}=1024^{10}<10\cdot1000^{10}$$

whence it follows that 2^{100} consists of less than 32 digits. Therefore the number 2^{100} has 31 digits.

Remark. This problem can easily be solved with the aid of the table of logarithms. Since $\log 2=0{,}30103$, we have $\log 2^{100}=100\log 2=30.103$, and consequently the decimal representation of the number 2^{100} involves 31 digits. However, in this problem it is required to obtain the result without using the table of logarithms.

212. (a) *First solution.* Let us denote by A the product $(1/2)(3/4)(5/6)\ldots(99/100)$ and let us also consider the product

$B = (2/3)(4/5)(6/7)\ldots(98/99)$. Since

$$\frac{2}{3} > \frac{1}{2}, \quad \frac{4}{5} > \frac{3}{4}, \quad \frac{6}{7} > \frac{5}{6}, \ldots, \quad \frac{98}{99} > \frac{97}{98}, \quad 1 > \frac{99}{100}$$

we have $B > A$. At the same time,

$$A \cdot B = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdots \frac{98}{99} \cdot \frac{99}{100} = \frac{1}{100}$$

It follows that

$$A^2 < AB = \frac{1}{100}, \quad \text{and therefore} \quad A < \frac{1}{10}$$

Further, $B < 2A = (3/4)(5/6)(7/8)\ldots(99/100)$ because

$$\frac{2}{3} < \frac{3}{4}, \quad \frac{4}{5} < \frac{5}{6}, \quad \frac{6}{7} < \frac{7}{8}, \ldots, \quad \frac{98}{99} < \frac{99}{100}$$

Consequently,

$$A \cdot 2A > AB = \frac{1}{100} \quad \text{and therefore} \quad A > \frac{1}{10\sqrt{2}}$$

Second solution. As before, we denote

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100} = A$$

Then

$$A^2 = \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdots \frac{99^2}{100^2}$$

whence

$$\frac{1^2}{2^2} \cdot \frac{3^2-1}{4^2} \cdot \frac{5^2-1}{6^2} \cdots \frac{99^2-1}{100^2} < A^2 < \frac{1^2}{2^2-1} \cdot \frac{3^2}{4^2-1} \cdot \frac{5^2}{6^2-1} \cdots \frac{99^2}{100^2-1}$$

Now let us use the formula $a^2 - b^2 = (a+b)(a-b)$ to factor the numerators of the fractions on the left-hand side and the denominators of the fractions on the right-hand side. This results in

$$\frac{1}{2 \cdot 2} \cdot \frac{2 \cdot 4}{4 \cdot 4} \cdot \frac{4 \cdot 6}{6 \cdot 6} \cdots \frac{98 \cdot 100}{100 \cdot 100} < A^2 < \frac{1}{1 \cdot 3} \cdot \frac{3 \cdot 3}{3 \cdot 5} \cdot \frac{5 \cdot 5}{5 \cdot 7} \cdots \frac{99 \cdot 99}{99 \cdot 101}$$

On cancelling the fractions we obtain

$$\frac{1}{200} < A^2 < \frac{1}{101}, \quad \text{whence} \quad \frac{1}{10\sqrt{2}} < A < \frac{1}{\sqrt{101}} < \frac{1}{10}$$

which is what we intended to prove.

Remark. In just the same way we can prove a more general relation

$$\frac{1}{2\sqrt{n}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}$$

(b) Let us prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}<\frac{1}{\sqrt{3n+1}}$$

for $n>1$.

The simplest way to prove this inequality is to use the method of mathematical induction. For $n=1$ we have

$$\frac{1}{2}=\frac{1}{\sqrt{3\cdot 1+1}}$$

Now let us suppose that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\leqslant\frac{1}{\sqrt{3n+1}}$$

for a certain value of n. On multiplying both members of the last inequality by $(2n+1)/(2n+2)$ we obtain

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2}\leqslant\frac{2n+1}{(2n+2)\sqrt{3n+1}}$$

On the other hand, we have

$$\left(\frac{2n+1}{(2n+2)\sqrt{3n+1}}\right)^2=\frac{(2n+1)^2}{12n^3+28n^2+20n+4}=$$
$$=\frac{(2n+1)^2}{(12n^3+28n^2+19n+4)+n}=\frac{(2n+1)^2}{(2n+1)^2(3n+4)+n}<\frac{1}{3n+4}$$

whence it follows that

$$\frac{2n+1}{(2n+2)\sqrt{3n+1}}<\frac{1}{\sqrt{3n+4}}$$

We thus obtain the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2}<\frac{1}{\sqrt{3(n+1)+1}}$$

By the principle of mathematical induction, it follows that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\leqslant\frac{1}{\sqrt{3n+1}}$$

for any n, the sign of equality appearing only in the case when $n=1$.

Now let us substitute $n=50$ into the last inequality; this results in the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{99}{100}<\frac{1}{\sqrt{3\cdot 50+1}}=\frac{1}{\sqrt{151}}=\frac{1}{12.288\ldots}$$

which is even stronger than the one we had to prove.

213. The solution of the problem is a consequence of the two inequalities

$$31^{11} < 32^{11} = (2^5)^{11} = 2^{5\cdot 11} = 2^{55}$$

and

$$17^{14} > 16^{14} = (2^4)^{14} = 2^{4\cdot 14} = 2^{56} > 2^{55}$$

which are quite evident; they imply that $31^{11} < 17^{14}$. (It should be noted that the decimal representations of 31^{11} and 17^{14} consist of very many digits, and it is rather difficult to compute these numbers.)

214. (a) Since $2^{2^2} = 2^4 = 16 < 27 = 3^3$, we obviously have

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{2^2}}}}}}}_{(n+1 \text{ digits})} < \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{3^3}}}}}}}_{(n \text{ digits})} \leqslant \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3^{3^3}}}}}}}_{(n \text{ digits})}$$

for any $n > 1$ (and, in particular, for $n = 1000$). (In the case when $n = 2$ the last inequality turns into an equality whereas the first inequality remains strict; there is only one case when the inequalities change their sense, namely when $n = 1$: it is clear that $4 = 2^2 > 3$.)

(b) Let us prove that the inequalities

$$\underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{(n-1 \text{ digits})} \leqslant \underbrace{2^{2^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}_{(n \text{ digits})} < \underbrace{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}_{(n \text{ digits})}$$

hold for any $n > 1$ (and, in particular, for $n = 1000$). Since the last inequality is evident, it only remains to prove the first one. The proof can easily be elaborated with the aid of the *method of mathematical induction*. It is clear that when $n = 2$, that is when $n - 2 = 0$, we have one and the same number $4 = 2^2$ on the left-hand and on the right-hand sides of the first inequality. Let us assume that the inequality has already been proved for a certain value of n and show that under this assumption the inequality remains true for the "next" value $n + 1 \geqslant 3$. We have

$$\underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}_{(n \text{ fours})} = (2^2)^{\overbrace{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}^{(n-1 \text{ fours})}} < (2^2)^{\overbrace{2^{2^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}^{(n-2 \text{ threes})}}$$

(here we use the induction hypothesis). Further, we have

$$(2^2)^{\overbrace{2^{2^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}^{(n-2 \text{ threes})}} = 2^{\overbrace{2\cdot 2^{2^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}^{(n-2 \text{ threes})}} = 2^{2^{\overbrace{(2+1)^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}^{(n-2 \text{ threes})}}} < \underbrace{2^{2^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}_{(n-1 \text{ threes})}$$

because it is evident that

$$\underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{(n-1\text{ threes})} > \underbrace{2^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}_{(n-1\text{ digits})} + 1$$

This argument completes the proof of the required inequality.

215. Let us denote by k the number of digits in the decimal representation of the number 1974^n; then $10^{k-1} \leqslant 1974^n < 10^k$. (Since $1974^n > 1000^n = 10^{3n}$, it is clear that $k \geqslant 3n$.) If the representation of the number $1974^n + 2^n$ contains more than k digits then $1974^n + 2^n \geqslant 10^k$. However, since $1974^n = 2^n \cdot 987^n$ and $1974^n + 2^n = 2^n\ (987^n + 1)$, we obtain (on cancelling the corresponding inequalities by 2^n) the relations

$$987^n < 2^{k-n} \cdot 5^k \quad \text{and} \quad 987^n + 1 \geqslant 2^{k-n} \cdot 5^k$$

which can hold simultaneously only when $987^n + 1 = 2^{k-n} \cdot 5^k$ (in this case $987^n = 2^{k-n} \cdot 5^k - 1$).

Since $k - n \geqslant 3n - n = 2n$, the number $2^{k-n} \cdot 5^n$ is multiple of 8 (and even of 16) for $n > 2$. On the other hand, the division of 987 by 8 leaves a remainder 3; therefore the division of 987^n by 8 leaves the same remainder as the division of 3^n by 8. Further, on raising consecutively 3 to the powers 1, 2, 3, ... and replacing every time the resultant power by the remainder obtained when that power is divided by 8, we see that the division of the powers of 3 by 8 leaves remainders forming an alternating sequence of the form 3, 1, 3, 1; 3, 1; ... . Therefore the division of the number $987^n + 1$ by 8 can only leave remainders equal to 4 and 2, and this number can never be exactly divisible by 8. We have thus arrived at a contradiction which proves the assertion stated in the problem.

216. Since the last digit of the number 36^m is 6 for any natural m and the last digit of the number 5^n is 5 for any natural n, the last digit of the difference $36^m - 5^n$ is 1 when $36^m > 5^n$ and the last digit of the difference $5^n - 36^m$ is 9 for $36^m < 5^n$. Therefore the last digit in the decimal representation of the number $N = |36^m - 5^n|$ can only be 1 or 9, and the smallest possible values of this number can be 1 or 9 or 11. For $m = 1$ and $n = 2$ we obviously have $N = 11$. Let us show that the equalities $N = 9$ and $N = 1$ are *impossible*; this will mean that it is the value $N = 11$ that is the *smallest* one. Indeed, if we had the equality $5^n - 36^m = 9$, it would follow that the number $5^n = 36^m + 9$ is multiple of 9, which is impossible; if we had the equality $36^m - 5^n = 1$, it would follow that $5^n = 36^m - 1 = 6^{2m} - 1 = (6^m + 1)(6^m - 1)$ or, which is the same, $6^m - 1 = 5^k$ and $6^m + 1 = 5^{n-k}$, which is impossible because the number $6^m + 1$ ends with the digit 7 and therefore cannot be equal to a power of 5. We have thus proved that the smallest value of the absolute value of $36^m - 5^n$ is equal to 11.

217. We have

$$\frac{1}{2^{100}} C(100, 50) = \frac{1\cdot 2\cdot 3 \dots 100}{2^{50}(1\cdot 2\cdot 3 \dots 50)\cdot 2^{50}(1\cdot 2\cdot 3 \dots 50)} =$$

$$= \frac{1\cdot 2\cdot 3 \dots 100}{(2\cdot 4\cdot 6 \dots 100)\cdot(2\cdot 4\cdot 6 \dots 100)} = \frac{1\cdot 3\cdot 5 \dots 99}{2\cdot 4\cdot 6 \dots 100}$$

and therefore it only remains to use the result established in the solution of Problem 212 (a).

218. It is required to find which of the two numbers $101^n - 99^n$ and 100^n is greater. Let us consider the ratio

$$\frac{101^n - 99^n}{100^n} = \frac{(100+1)^n - (100-1)^n}{100^n} =$$

$$= \frac{2\left(C(n, 1)\cdot 100^{n-1} + C(n, 3)\cdot 100^{n-3} + \dots\right)}{100^n} =$$

$$= 2\left(\frac{n}{100} + \frac{n(n-1)(n-2)}{3!\cdot 100^3} + \dots\right)$$

It readily follows that for $n \geqslant 50$ this ratio exceeds 1. Let us show that for $n = 49$ this ratio also exceeds 1:

$$2\left(\frac{49}{100} + \frac{49\cdot 48\cdot 47}{3!\cdot 100^3} + \dots\right) > 2\left(\frac{49}{100} + \frac{18\,424}{100^3}\right) > 2\left(\frac{49}{100} + \frac{100^2}{100^3}\right) = 1$$

Now let us show that for $n = 48$ this ratio is less than 1. Indeed,

$$2\left(\frac{48}{100} + \frac{48\cdot 47\cdot 46}{3!\cdot 100^3} + \frac{48\cdot 47\cdot 46\cdot 45\cdot 44}{5!\cdot 100^5} + \dots\right) <$$

$$< 2\left(\frac{48}{100} + \frac{48^3}{(1\cdot 2\cdot 3)\cdot 100^3} + \frac{48^5}{(1\cdot 2\cdot 3)(2\cdot 3)\,100^5} + \right.$$

$$\left. + \frac{48^7}{(1\cdot 2\cdot 3)(2\cdot 3)(2\cdot 3)\,100^7} + \dots\right) =$$

$$= 2\left(\frac{48}{100} + \frac{1}{6}\left(\frac{48}{100}\right)^3 + \frac{1}{6^2}\left(\frac{48}{100}\right)^5 + \dots\right) <$$

$$< 2\,\frac{\frac{48}{100}}{1 - \frac{1}{6}\left(\frac{48}{100}\right)^2} = \frac{9600}{9616} < 1$$

For $n < 48$ this ratio is of course also less than 1.

Thus, finally, the number $99^n + 100^n$ is greater than 101^n for $n \leqslant 48$ and is less than 101^n for $n > 48$.

219. Let us begin with proving the following auxiliary proposition: a product of n consecutive whole numbers is greater than the square root of the nth power of the product of the smallest of these numbers by the greatest one. If we denote these numbers as $a, a+1, \ldots, a+n-1$ then the kth number, counting from left to right, is equal to $a+k-1$, and the kth number, counting from right to left, is equal to $a+n-k$. The product of these numbers satisfies the relation

$$(a+k-1)(a+n-k)=a^2+an-a+(k-1)(n-k)\geqslant$$
$$\geqslant a^2+an-a=a(a+n-1)$$

where the sign of equality can only appear when $k=1$ or $k=n$. In other words, the product of two numbers of the form $a+k-1$ and $a+n-k$ (in the case when n is odd these two numbers may coincide with the number at the middle of the sequence $a, a+1, \ldots, a+n-k$) always exceeds the product of the extreme numbers. It follows that for the product of all numbers $a, a+1, \ldots, a+n-1$ we have

$$a(a+1)\ldots(a+n-1)\geqslant[a(a+n-1)]^{\frac{n}{2}}=[\sqrt{a(a+n-1)}]^n$$

where the sign of equality occurs only for $n=1$ or $n=2$.

Now let us prove that $300! > 100^{300}$. We have

$$1\cdot 2\ldots 25>\sqrt{25^{25}}=5^{25}$$
$$26\ldots 50>(\sqrt{26\cdot 50})^{25}>35^{25}$$
$$51\ldots 100>(\sqrt{51\cdot 100})^{50}>70^{50}$$
$$101\ldots 200>\sqrt{100^{100}}\cdot\sqrt{200^{100}}=10^{200}\cdot 2^{50}$$

and

$$201\ldots 300>\sqrt{200^{100}}\cdot\sqrt{300^{100}}=10^{200}\cdot 2^{50}\cdot 3^{50}$$

On multiplying the left-hand and the right-hand members of the inequalities we obtain

$$300!>5^{25}\cdot 35^{25}\cdot 70^{50}\cdot 10^{400}\cdot 2^{100}\cdot 3^{50}=$$
$$=5^{50}\cdot 7^{25}\cdot 5^{50}\cdot 14^{50}\cdot 10^{400}\cdot 2^{100}\cdot 3^{50}=10^{500}\cdot 7^{25}\cdot 14^{50}\cdot 3^{50}=$$
$$=10^{500}\cdot 21^{25}\cdot 42^{25}\cdot 14^{25}>10^{500}\cdot 20^{25}\cdot 40^{25}\cdot 14^{25}=$$
$$=10^{550}\cdot 2^{25}\cdot 4^{25}\cdot 14^{25}=10^{550}\cdot 112^{25}=10^{600}\cdot 1.12^{25}>10^{600}=100^{300}$$

Remark. A more general result is stated in Problem 223.

220. Let us prove that

$$1+\frac{k}{n}\leqslant\left(1+\frac{1}{n}\right)^k<1+\frac{k}{n}+\frac{k^2}{n^2}$$

for any positive integer k such that $k \leqslant n$. To this end we shall use the method of mathematical induction. For $k = 1$ the required relation obviously holds. Now let us assume that it holds for some k and prove that under this assumption it holds for $k+1$ as well. We have

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)^{k}\left(1+\frac{1}{n}\right) \geqslant \left(1+\frac{k}{n}\right)\left(1+\frac{1}{n}\right) = \\ = 1 + \frac{k+1}{n} + \frac{k}{n^2} > 1 + \frac{k+1}{n}$$

It should be noted that here we have not used the relation $k \leqslant n$, and consequently this inequality holds for any positive integer k. Now let us put $k \leqslant n$; then we obtain

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)^{k}\left(1+\frac{1}{n}\right) < \left(1+\frac{k}{n}+\frac{k^2}{n^2}\right)\left(1+\frac{1}{n}\right) = \\ = 1 + \frac{k+1}{n} + \frac{k^2+2k+1}{n^2} - \frac{k+1}{n^2} + \frac{k^2}{n^3} = \\ = 1 + \frac{k+1}{n} + \frac{(k+1)^2}{n^2} - \frac{n(k+1)-k^2}{n^3} < 1 + \frac{k+1}{n} + \frac{(k+1)^2}{n^2}$$

because $n(k+1) > k^2$ for $n \geqslant k$.

On substituting the value $k = n$ into the inequalities we have derived we obtain

$$2 = 1 + \frac{n}{n} \leqslant \left(1+\frac{1}{n}\right)^{n} < 1 + \frac{n}{n} + \frac{n^2}{n^2} = 3$$

221. By virtue of the result established in the solution of Problem 220, we have

$$(1.000001)^{1\,000\,000} = \left(1 + \frac{1}{1\,000\,000}\right)^{1\,000\,000} > 2$$

222. We obviously have

$$\frac{(1001)^{999}}{(1000)^{1000}} = \left(\frac{1001}{1000}\right)^{1000} \cdot \frac{1}{1001} = \left(1+\frac{1}{1000}\right)^{1000} \cdot \frac{1}{1001} < 3 \cdot \frac{1}{1001} < 1$$

(see Problem 220), and consequently

$$1000^{1000} > 1001^{999}$$

223. Let us suppose that the inequalities indicated in the condition of the problem hold for some n. To prove that they hold for $n+1$ as well it is sufficient to verify the validity of the following inequalities:

$$\left(\frac{n+1}{2}\right)^{n+1} : \left(\frac{n}{2}\right)^{n} \geqslant n+1 \geqslant \left(\frac{n+1}{3}\right)^{n+1} : \left(\frac{n}{3}\right)^{n}$$

On cancelling these inequalities by $n+1$ we obtain the equivalent inequalities $(1+1/n)^n/2 \geqslant 1 \geqslant (1+1/n)^n/3$ which follow from the inequality $2 \leqslant (1+1/n)^n < 3$.

Now it only remains to note that the assertion of the problem holds for $n=6$ because

$$\left(\frac{6}{2}\right)^6 = 3^6 = 729, \quad 6! = 720 \quad \text{and} \quad \left(\frac{6}{3}\right)^6 = 2^6 = 64$$

224. **(a)** By Newton's binomial formula, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + C(n, 1)\frac{1}{n} + C(n, 2)\frac{1}{n^2} + \ldots + C(n, n-1)\frac{1}{n^{n-1}} + \frac{1}{n^n} =$$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} + \frac{1}{n^3}\ldots$$

$$\ldots + \frac{n(n-1)\ldots 2}{(n-1)!}\frac{1}{n^{n-1}} + \frac{n(n-1)\ldots 1}{n!}\frac{1}{n^n} =$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots$$

$$\ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

and, similarly,

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) +$$

$$+ \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \ldots$$

$$\ldots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n-1}{n+1}\right) +$$

$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n-1}{n+1}\right)\left(1-\frac{n}{n+1}\right)$$

The comparison of these expressions shows that $(1+1/(n+1))^{n+1} > (1+1/n)^n$, whence follows the assertion stated in the problem.

(b) Let us consider the ratio

$$\left(1+\frac{1}{n}\right)^{n+1} : \left(1+\frac{1}{n-1}\right)^n = \left(\frac{n+1}{n}\right)^{n+1} : \left(\frac{n}{n-1}\right)^n =$$

$$= \frac{(n+1)^{n+1}(n-1)^n}{n^{2n+1}} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \frac{n+1}{n} = \left(1-\frac{1}{n^2}\right)^n\left(1+\frac{1}{n}\right)$$

For $n \geqslant 2$ we have

$$\left(1-\frac{1}{n^2}\right)^n = 1 - n\cdot\frac{1}{n^2} + \frac{n(n-1)}{2!}\frac{1}{n^4} - \frac{n(n-1)(n-2)}{3!}\frac{1}{n^6} + \\ + \frac{n(n-1)(n-2)(n-3)}{4!}\frac{1}{n^8} - \ldots = \\ = 1 - \frac{1}{n} + \frac{1}{2}\frac{n-1}{n^3} - \left[\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{n^3} - \right. \\ \left. - \frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{1}{n^4}\right] - \ldots \leqslant \\ \leqslant \ldots 1 - \frac{1}{n} + \frac{1}{2}\frac{1}{n^2} - \frac{1}{2}\frac{1}{n^3}$$

On the other hand,

$$\left(1-\frac{1}{n}+\frac{1}{2}\frac{1}{n^2}-\frac{1}{2}\frac{1}{n^3}\right)\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2}\frac{1}{n^2} - \frac{1}{2}\frac{1}{n^4} < 1$$

Consequently, $(1 - 1/n^2)^n(1 + 1/n) < 1$, and therefore

$$\left(1+\frac{1}{n}\right)^{n+1} : \left(1+\frac{1}{n-1}\right)^n < 1,$$

that is

$$\left(1+\frac{1}{n}\right)^{n+1} < \left(1+\frac{1}{n-1}\right)^n$$

whence follows the assertion we had to prove.

225. We shall use the *proof by induction.*

1°. Let us show that

$$n! > \left(\frac{n}{e}\right)^n \qquad (*)$$

for any positive integer n. Indeed, for $n = 1$ this inequality obviously holds: $1! = 1 > 1/e$. Now we assume that inequality (*) has already been proved for certain n and then show that under this assumption it holds for $n + 1$ as well; in other words, we must establish the inequality

$$(n+1)! > \left(\frac{n+1}{e}\right)^{n+1}$$

By virtue of the result obtained in the solution of Problem 224 (a), we have

$$e > \left(1+\frac{1}{n}\right)^n, \quad \text{that is} \quad \frac{e}{\left(1+\frac{1}{n}\right)^n} > 1$$

Using inequality (*) we now find

$$(n+1)! = (n+1)\,n! > \left(\frac{n}{e}\right)^n (n+1) = \left(\frac{n+1}{e}\right)^{n+1}\frac{n^n e}{(n+1)^n} = \\ = \left(\frac{n+1}{e}\right)^{n+1}\frac{e}{\left(1+\frac{1}{n}\right)^n} > \left(\frac{n+1}{e}\right)^{n+1}$$

By the principle of mathematical induction, it follows that inequality (*) is fulfilled for any positive integer n.

2°. Now we pass to the inequality

$$n! < n\left(\frac{n}{e}\right)^n \qquad (**)$$

We have to prove that it holds for all integral values of n exceeding 6. Using tables of logarithms (tables of natural logarithms are particularly convenient for this purpose) we easily check that inequality (**) holds for $n = 7$;

$$7! < 7\left(\frac{7}{e}\right)^7$$

This means that $6! < (7/e)^7$ because $\ln 6! = \ln 720 \approx 6.58$ and

$$\ln\left(\frac{7}{e}\right)^7 = 7(\ln 7 - 1) \approx 6.62$$

Now let us assume that inequality (**) has already been proved for a certain n. By virtue of the result of Problem 224 (b), we have

$$\left(1+\frac{1}{n}\right)^{n+1} > e, \quad \text{that is} \quad \frac{e}{\left(1+\frac{1}{n}\right)^{n+1}} < 1$$

From inequality (**) we now derive

$$(n+1)! = (n+1)\,n! < (n+1)\,n\left(\frac{n}{e}\right)^n =$$

$$= (n+1)\left(\frac{n+1}{e}\right)^{n+1} \frac{n^{n+1}e}{(n+1)^{n+1}} =$$

$$= (n+1)\left(\frac{n+1}{e}\right)^{n+1} \frac{e}{\left(1+\frac{1}{n}\right)^{n+1}} < (n+1)\left(\frac{n+1}{e}\right)^{n+1}$$

We see that inequality (**) with n replaced by $n+1$ also holds. Since inequality (**) holds for $n = 7$, by the principle of mathematical induction it follows that (**) holds for any integer n greater than 6 as well, which is what we intended to prove.

226. We shall proceed from the fact that for $x > 1$ the greatest term in the sum $S = x^k + x^{k-1} + x^{k-2} + \ldots + x + 1$ is the first one and the smallest term is the last one while, conversely, for $x < 1$ the greatest term is the last one and the smallest term is the first one. It follows that

$$(k+1)\,x^k > S > k+1 \quad \text{for} \quad x > 1$$

and

$$(k+1)x^k < S < k+1 \quad \text{for} \quad x < 1$$

On multiplying both members of each of these inequalities by $x-1$ we see that

$$(k+1)x^k(x-1) > x^{k+1} - 1 > (k+1)(x-1)$$

for $x \neq 1$. Let us put $x = p/(p-1)$ in the last inequality; this results in

$$\frac{(k+1)p^k}{(p-1)^{k+1}} > \frac{p^{k+1}-(p-1)^{k+1}}{(p-1)^{k+1}} > \frac{(k+1)(p-1)^k}{(p-1)^{k+1}}$$

Similarly, on putting $x = (p+1)/p$ we obtain

$$\frac{(k+1)(p+1)^k}{p^{k+1}} > \frac{(p+1)^{k+1}-p^{k+1}}{p^{k+1}} > \frac{(k+1)p^k}{p^{k+1}}$$

Now it follows that

$$(p+1)^{k+1} - p^{k+1} > (k+1)p^k > p^{k+1} - (p-1)^{k+1}$$

Next we consecutively put $p = 1, 2, 3, \ldots, n$ in the last relation to obtain

$$2^{k+1} - 1^{k+1} > (k+1)1^k > 1^{k+1} - 0$$
$$3^{k+1} - 2^{k+1} > (k+1)2^k > 2^{k+1} - 1^{k+1}$$
$$4^{k+1} - 3^{k+1} > (k+1)3^k > 3^{k+1} - 2^{k+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(n+1)^{k+1} - n^{k+1} > (k+1)n^k > n^{k+1} - (n-1)^{k+1}$$

On adding together all these inequalities we find

$$(n+1)^{k+1} - 1 > (k+1)(1^k + 2^k + 3^k + \ldots + n^k) > n^{k+1}$$

Finally, the division of all the members of the last inequalities by $k+1$ yields

$$\left[\left(1+\frac{1}{n}\right)^{k+1} - \frac{1}{n^{k+1}}\right]\frac{1}{k+1}n^{k+1} >$$
$$> 1^k + 2^k + 3^k + \ldots + n^k > \frac{1}{k+1}n^{k+1}$$

whence follows the inequality we intended to prove.

227. (a) It is quite evident that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n}}_{n \text{ times}} = \frac{1}{2}$$

On the other hand, we have

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ \dots\ +\frac{1}{2n}=\frac{1}{2}\Big[\Big(\frac{1}{n}+\frac{1}{2n}\Big)+$$
$$+\Big(\frac{1}{n+1}+\frac{1}{2n-1}\Big)+\Big(\frac{1}{n+2}+\frac{1}{2n-2}\Big)+\ \dots\ +\Big(\frac{1}{2n}+\frac{1}{n}\Big)\Big]=$$
$$=\frac{1}{2}\Big[\frac{3n}{2n^2}+\frac{3n}{2n^2+(n-1)}+\frac{3n}{2n^2+2(n-2)}+\ \dots\ +\frac{3n}{2n^2}\Big]<$$
$$<\frac{1}{2}\Big[\underbrace{\frac{3n}{2n^2}+\frac{3n}{2n^2}+\dots+\frac{3n}{2n^2}}_{n+1\text{ times}}\Big]=\frac{1}{2}(n+1)\frac{3}{2n}=\frac{3}{4}+\frac{3}{4n}<\frac{3}{4}+\frac{1}{n}$$

This relation implies the second assertion stated in the condition of the problem.

(b) We begin with the obvious relation

$$\frac{1}{3n}+\frac{1}{3n+1}<\frac{1}{2n}+\frac{1}{2n}=\frac{1}{n}$$

Now it follows that

$$\frac{1}{n+1}+\frac{1}{n+2}+\ \dots\ +\frac{1}{3n-1}+\Big(\frac{1}{3n}+\frac{1}{3n+1}\Big)<$$
$$<\underbrace{\frac{1}{n}+\frac{1}{n}+\ \dots\ +\frac{1}{n}}_{2n-1\text{ times}}+\frac{1}{n}=\frac{2n}{n}=2$$

On the other hand, we have

$$\frac{1}{n+1}+\frac{1}{n+2}+\ \dots\ +\frac{1}{3n+1}=\frac{1}{2}\Big[\Big(\frac{1}{n+1}+\frac{1}{3n+1}\Big)+$$
$$+\Big(\frac{1}{n+2}+\frac{1}{3n}\Big)+\Big(\frac{1}{n+3}+\frac{1}{3n-1}\Big)+\ \dots\ +\Big(\frac{1}{3n+1}+\frac{1}{n+1}\Big)\Big]=$$
$$=\frac{1}{2}\Big[\frac{4n+2}{(2n+1)^2-n^2}+\frac{4n+2}{(2n+1)^2-(n-1)^2}+$$
$$+\frac{4n+2}{(2n+1)^2-(n-2)^2}+\ \dots\ +\frac{4n+2}{(2n+1)^2-n^2}\Big]>$$
$$>\frac{1}{2}\Big[\underbrace{\frac{4n+2}{(2n+1)^2}+\frac{4n+2}{(2n+1)^2}+\ \dots\ +\frac{4n+2}{(2n+1)^2}}_{2n+1\text{ times}}\Big]=$$
$$=\frac{1}{2}(2n+1)\frac{4n+2}{(2n+1)^2}=1$$

228. **(a)** First we shall prove the inequalities

$$2\sqrt{n+1}-2\sqrt{n}<\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{n-1}$$

Indeed,

$$2\sqrt{n+1}-2\sqrt{n}=\frac{2(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}=$$
$$=\frac{2}{\sqrt{n+1}+\sqrt{n}}<\frac{2}{\sqrt{n}+\sqrt{n}}=\frac{1}{\sqrt{n}}$$

and the second inequality is proved in a similar way.

Now we can write

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{1\,000\,000}}>1+2[(\sqrt{3}-\sqrt{2})+$$
$$+(\sqrt{4}-\sqrt{3})+\ldots+(\sqrt{1\,000\,001}-\sqrt{1\,000\,000})]=$$
$$=1+2(\sqrt{1\,000\,001}-\sqrt{2})>2\cdot 1001-\sqrt{8}+$$
$$+1>2000-3+1=1998$$

Analogously,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{1\,000\,000}}<1+2[(\sqrt{2}-1)+$$
$$+(\sqrt{3}-\sqrt{2})+\ldots+(\sqrt{1\,000\,000}-\sqrt{999\,999})]=$$
$$=1+2(\sqrt{1\,000\,000}-1)=1+2\cdot 999=1999$$

Consequently, the integral part of the sum $1+1/\sqrt{2}+1/\sqrt{3}+\ldots$ $\ldots+1/\sqrt{1000\,000}$ is equal to 1998.

(b) By complete analogy with the solution of Problem 228 (a), we obtain

$$\frac{1}{\sqrt{10\,000}}+\frac{1}{\sqrt{10\,001}}+\ldots+\frac{1}{\sqrt{1\,000\,000}}>$$
$$>2[(\sqrt{10\,001}-\sqrt{10\,000})+(\sqrt{10\,002}-\sqrt{10\,001})+\ldots$$
$$\ldots+(\sqrt{1\,000\,001}-\sqrt{1\,000\,000})]=$$
$$=2(\sqrt{1\,000\,001}-\sqrt{10\,000})>2(1000-100)=1800$$

and

$$\frac{1}{\sqrt{10\,000}}+\frac{1}{\sqrt{10\,001}}+\ldots+\frac{1}{\sqrt{1\,000\,000}}<$$
$$<2[(\sqrt{10\,000}-\sqrt{9999})+(\sqrt{10\,001}-\sqrt{10\,000})+\ldots$$
$$\ldots+(\sqrt{1\,000\,000}-\sqrt{999\,999})]=$$
$$=2(\sqrt{1\,000\,000}-\sqrt{9999})=$$
$$=2000-\sqrt{39\,996}<2000-199.98=1800.02$$

Consequently, the sum $1/\sqrt{10\,000}+1/\sqrt{10\,001}+\ldots$ $\ldots+1/\sqrt{1\,000\,000}$ is equal to 1800 with an accuracy of 0.02.

229. First of all we note that the comparison of the two relations

$$\left(1+\frac{1}{n}\right)^2=1+2\frac{1}{n}+\frac{1}{n^2}$$

and

$$\left(1+\frac{2}{3}\frac{1}{n}\right)^3=1+2\frac{1}{n}+\frac{4}{3}\frac{1}{n^2}+\frac{8}{27}\frac{1}{n^3}$$

shows that

$$\left(1+\frac{2}{3}\frac{1}{n}\right)^3>\left(1+\frac{1}{n}\right)^2$$

for any positive integer n. It follows that $1+2/3n>(1+1/n)^{2/3}$. On multiplying the last inequality by $n^{2/3}$ we obtain $n^{2/3}+2n^{-1/3}/3>(n+1)^{2/3}$ whence, finally,

$$\frac{1}{\sqrt[3]{n}}>\frac{3}{2}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}\right]$$

Similarly,

$$\left(1-\frac{2}{3}\frac{1}{n}\right)^3=1-2\frac{1}{n}+\frac{4}{3}\frac{1}{n^2}-\frac{8}{27}\frac{1}{n^3}>1-2\frac{1}{n}+\frac{1}{n^2}=\left(1-\frac{1}{n}\right)^2$$

(because $1/3n^2-8/27n^3>1/3n^2-1/3n^3\geqslant 0$) whence it follows that

$$1-\frac{2}{3}\frac{1}{n}>\left(1-\frac{1}{n}\right)^{\frac{2}{3}},\quad n^{\frac{2}{3}}-\frac{2}{3}n^{-\frac{1}{3}}>(n-1)^{\frac{2}{3}}\ \text{and}$$

$$\frac{1}{\sqrt[3]{n}}<\frac{3}{2}\left[\sqrt[3]{n^2}-\sqrt[3]{(n-1)^2}\right]$$

Now we can write

$$\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{5}}+\ldots+\frac{1}{\sqrt[3]{1\,000\,000}}>$$
$$>\frac{3}{2}\left[\left(\sqrt[3]{5^2}-\sqrt[3]{4^2}\right)+\left(\sqrt[3]{6^2}-\sqrt[3]{5^2}\right)+\ldots\right.$$
$$\left.\ldots+\left(\sqrt[3]{1\,000\,001^2}-\sqrt[3]{1\,000\,000^2}\right)\right]=$$
$$=\frac{3}{2}\left(\sqrt[3]{1\,000\,002\,000\,001}-\sqrt[3]{16}\right)>\frac{3}{2}\cdot 10\,000-\sqrt[3]{54}>$$
$$>15\,000>4=14\,996$$

On the other hand,

$$\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{5}}+\ldots+\frac{1}{\sqrt[3]{1\,000\,000}}<$$
$$<\frac{3}{2}\left[\left(\sqrt[3]{4^2}-\sqrt[3]{3^2}\right)+\left(\sqrt[3]{5^2}-\sqrt[3]{4^2}\right)+\ldots\right.$$
$$\left.\ldots+\left(\sqrt[3]{1\,000\,000^2}-\sqrt[3]{999\,999^2}\right)\right]=$$
$$=\frac{3}{2}\left(\sqrt[3]{1\,000\,000\,000\,000}-\sqrt[3]{9}\right)<\frac{3}{2}(10\,000-2)=14\,997$$

Thus, the integral part of the sum $1/\sqrt[3]{4}+1/\sqrt[3]{5}+\ldots$ $\ldots+1/\sqrt[3]{1\,000\,000}$ is equal to 14 996.

230. (a) We obviously have

$$\frac{1}{10^2}+\frac{1}{11^2}+\ldots+\frac{1}{1000^2}>\frac{1}{10\cdot 11}+\frac{1}{11\cdot 12}+\ldots+\frac{1}{1000\cdot 1001}=$$
$$=\left(\frac{1}{10}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{12}\right)+\ldots+\left(\frac{1}{1000}-\frac{1}{1001}\right)=$$
$$=\frac{1}{10}-\frac{1}{1001}>0.1-0.001=0.099$$

and, similarly,

$$\frac{1}{10^2}+\frac{1}{11^2}+\ldots+\frac{1}{1000^2}<\frac{1}{9\cdot 10}+\frac{1}{10\cdot 11}+\ldots+\frac{1}{999\cdot 1000}=$$
$$=\left(\frac{1}{9}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{11}\right)+\ldots+\left(\frac{1}{999}-\frac{1}{1000}\right)=$$
$$=\frac{1}{9}-\frac{1}{1000}<0.112-0.001=0.111$$

Consequently, the sum $1/10^2+1/11^2+\ldots+1/1000^2$ is equal to 0.105 with an accuracy of 0.006.

(b) First of all we note that

$$\frac{1}{10!}+\frac{1}{11!}+\frac{1}{12!}+\ldots+\frac{1}{1000!}>\frac{1}{10!}=\frac{1}{3\,628\,800}\approx 0.000000275$$

On the other hand,

$$\frac{1}{10!}+\frac{1}{11!}+\frac{1}{12!}+\ldots+\frac{1}{1000!}<\frac{1}{9}\left\{\frac{9}{10!}+\frac{10}{11!}+\frac{11}{12!}+\ldots+\frac{999}{1000!}\right\}=$$
$$=\frac{1}{9}\left\{\frac{10-1}{10!}+\frac{11-1}{11!}+\frac{12-1}{12!}+\ldots+\frac{1000-1}{1000!}\right\}=$$
$$=\frac{1}{9}\left\{\frac{1}{9!}-\frac{1}{10!}+\frac{1}{10!}-\frac{1}{11!}+\frac{1}{11!}-\frac{1}{12!}+\ldots+\frac{1}{999!}-\frac{1}{1000!}\right\}=$$
$$=\frac{1}{9}\left(\frac{1}{9!}-\frac{1}{1000!}\right)<\frac{1}{9}\cdot\frac{1}{9!}=\frac{1}{3\,265\,920}\approx 0.000000305$$

Thus, the sum $1/10!+1/11!+\ldots+1/1000!$ is equal to 0.00000029 with an accuracy of 0.000000015.

231. Let us prove that the sum

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n}$$

can be made greater than any preassigned number N by taking a sufficiently large value of n. Assuming that N is an integer (this does not lead to loss of generality) we take $n = 2^{2N}$; then

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-1}+\frac{1}{n}=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+$$
$$+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots+\left(\frac{1}{2^{2N-1}+1}+\frac{1}{2^{2N-1}+2}+\dots\right.$$
$$\left.\dots+\frac{1}{2^{2N}-1}+\frac{1}{2^{2N}}\right)>1+\underbrace{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}}_{2N\text{ times}}>N+1$$

(according to the result established in the solution of Problem 227 (a), each of the sums in the parentheses exceeds 1/2).

Remark. The assertion of this problem can also be proved on the basis of the result established in the solution of Problem 227 (b).

232. Let us denote by n_k the number of those summands lying between $1/10^k$ and $1/10^{k+1}$ (including the number $1/10^k$ but not $1/10^{k+1}$) which are not deleted. If a summand $1/q$ lying between $1/10^{k-1}$ and $1/10^k$ is not deleted then among the summands $1/10q$, $1/(10q+1)$, $1/(10q+2)$, $\dots$, $1/(10q+8)$, $1/(10q+9)$ lying between $1/10^k$ and $1/10^{k+1}$ only the last one is deleted. In case the summand $1/q$ is deleted all the summands $1/10q$, $1/(10q+1)$, $\dots$, $1/(10q+8)$, $1/(10q+9)$ are also deleted. It follows that

$$n_k = 9n_{k-1}$$

Since $n_0 = 8$ (because among the summands 1, 1/2, 1/3, $\dots$, 1/8, 1/9 only 1/9 is deleted), we have

$$n_1 = 8\cdot 9 = 72,\quad n_2 = 8\cdot 9^2,\quad \dots,\quad n_k = 8\cdot 9^k$$

Now let us take the sum $1+1/2+1/3+\dots+1/n$ with $n < 10^{m+1}$. On adding to this sum the terms needed to obtain the sum $1+1/2+1/3+\dots+1/(10^{m+1}-1)$ and deleting those summands whose denominators involve 9 we can group the remaining terms thus:

$$\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{8}\right)+$$
$$+\left(\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\dots+\frac{1}{18}+\frac{1}{20}+\dots+\frac{1}{88}\right)+$$
$$+\left(\frac{1}{100}+\frac{1}{101}+\dots+\frac{1}{888}\right)+\dots+\left(\frac{1}{10^m}+\dots+\frac{1}{\underbrace{88\dots8}_{m+1\text{ eights}}}\right)$$

The last expression does not exceed the sum

$$1\cdot n_0+\frac{1}{10}\cdot n_1+\frac{1}{100}\cdot n_2+\dots+\frac{1}{10^{m-1}}\cdot n_{m-1}+\frac{1}{10^m}\cdot n_m$$

because it is obtained from that expression by replacing every sum in the parentheses by the product of the greatest of the expressions in the parentheses by the number of these expressions. Further, we obviously have

$$1 \cdot n_0 + \frac{1}{10} \cdot n_1 + \frac{1}{100} \cdot n_2 + \ldots + \frac{1}{10^{m-1}} \cdot n_{m-1} + \frac{1}{10^m} \cdot n_m =$$

$$= 8\left(1 + \frac{9}{10} + \frac{9^2}{10^2} + \ldots + \frac{9^{m-1}}{10^{m-1}} + \frac{9^m}{10^m}\right) =$$

$$= 8 \cdot \frac{1 - \frac{9^{m+1}}{10^{m+1}}}{1 - \frac{9}{10}} < 8 \cdot \frac{1}{1 - \frac{9}{10}} = 8 \cdot 10 = 80$$

whence follows the assertion we had to prove.

233. **(a)** Let us consider the sum $1 + 1/4 + 1/9 + \ldots + 1/n^2$ with n smaller than 2^{k+1} and also the sum $1 + 1/2^2 + 1/3^2 + \ldots + 1/(2^{k+1} - 1)^2$. On grouping the terms by analogy with the solution of Problem 231 we obtain

$$1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \ldots$$

$$\ldots + \left(\frac{1}{(2^k)^2} + \frac{1}{(2^k + 1)^2} + \ldots + \frac{1}{(2^{k+1} - 1)^2}\right) < 1 + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) +$$

$$+ \left(\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}\right) + \ldots$$

$$\ldots + \left(\frac{1}{(2^k)^2} + \frac{1}{(2^k)^2} + \ldots + \frac{1}{(2^k)^2}\right) =$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} = \frac{1 - \frac{1}{2^{k+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^k} < 2$$

which is what we intended to prove.

Remark. In a completely similar manner we can show that if α is a number greater than 1 then

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha} < \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}$$

for any n.

Thus, for any $\alpha > 1$ the sum

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha}$$

remains bounded for arbitrarily large n (from the result of Problem 231 it follows that for $\alpha \leqslant 1$ the sum $1 + 1/2^\alpha + 1/3^\alpha + \ldots + 1/n^\alpha$ can be made arbitrarily large by taking a sufficiently large value of n).

(b) We obviously have

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}<\left(\frac{1}{1\cdot 2}-\frac{1}{4}\right)+$$
$$+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{(n-1)\,n}=$$
$$=\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(n-1)\,n}\right)-\frac{1}{4}$$

Further, since $1/(k-1)k=1/(k-1)-1/k$ for all $k=2, 3, \ldots, n$, there holds the equality

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(n-1)\,n}=1-\frac{1}{n}<1$$

and, consequently,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}<1+\left(1-\frac{1}{4}\right)=1\frac{3}{4}$$

which is what we had to prove.

234. We shall first prove the inequality

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}<$$
$$<\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^k}\right)\left(1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^k}\right)\times\ldots$$
$$\ldots\times\left(1+\frac{1}{p_l}+\frac{1}{p_l^2}+\ldots+\frac{1}{p_l^k}\right)$$

where k is an integer such that $2^k \leqslant n < 2^{k+1}$ and p_l is the greatest prime number not exceeding n. To prove the inequality we open the parentheses on the right-hand side. *Every* integral number m from 1 to n can be represented as a product of powers of the prime numbers 1, 3, 5, ..., p_l in the form

$$m=2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\ldots p_l^{\alpha_l}$$

where all the exponents $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_l$ are nonnegative integers (which, of course, do not exceed k). Therefore the sum obtained after the parentheses have been opened on the right-hand side involves a summand equal to $1/m$ which is the product of the numbers $1/2^{\alpha_1}$, $1/3^{\alpha_2}$, $1/5^{\alpha_3}$ etc. taken from the first expression in the parentheses, from the second expression in the parentheses, from the third expression in the parentheses etc. respectively. Hence, after the parentheses have been opened, the sum on the right-hand side involves *all* the summands 1, 1/2, 1/3, 1/4, ..., $1/(n-1)$, $1/n$ and some other positive summands. This means that the right-hand member of the given inequality is in fact greater than the left-hand member.

Now, taking logarithms of both members we obtain

$$\log\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}\right)<$$

$$<\log\left[\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^k}\right)\left(1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^k}\right)\times\ldots\right.$$

$$\left.\ldots\times\left(1+\frac{1}{p_l}+\frac{1}{p_l^2}+\ldots+\frac{1}{p_l^k}\right)\right]=$$

$$=\log\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^k}\right)+$$

$$+\log\left(1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^k}\right)+\ldots$$

$$\ldots+\log\left(1+\frac{1}{p_l}+\frac{1}{p_l^2}+\ldots+\frac{1}{p_l^k}\right)$$

Further, for any positive integers k and $p \geqslant 2$ we have

$$\log\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^k}\right)<\frac{2\log 3}{p}$$

Indeed,

$$1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^k}=\frac{1-\frac{1}{p^{k+1}}}{1-\frac{1}{p}}<\frac{1}{1-\frac{1}{p}}=\frac{p}{p-1}=1+\frac{1}{p-1}$$

and from the result established in Problem 220 it follows that

$$\left(1+\frac{1}{p-1}\right)^{p-1}<3,\quad 1+\frac{1}{p-1}<\sqrt[p-1]{3},\quad \log\left(1+\frac{1}{p-1}\right)<\frac{\log 3}{p-1}$$

and, besides, there obviously holds the inequality

$$\frac{2\log 3}{p}>\frac{\log 3}{p-1}$$

We thus conclude that

$$\log\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)<\frac{2\log 3}{2}+\frac{2\log 3}{3}+\frac{2\log 3}{5}+\ldots+\frac{2\log 3}{p_l}=$$

$$=2\log 3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p_l}\right)$$

If there existed a number N such that the sum $1+1/2+1/3+1/5+\ldots+1/p_l$ were less than N for any positive integer l

then for any positive integer n the inequalities

$$\log\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}\right)<$$
$$<2\log 3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p_l}\right)<2(N-1)\log 3$$

would be fulfilled. Therefore on taking exponentials of the leftmost and the rightmost members of the last inequalities we would obtain

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}<3^{2(N-1)}=N_1$$

where N_1 is independent of n. However, as was shown in the solution of Problem 231, such a number N_1 does not exist; consequently a number N such that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p_l}<N$$

for any positive integer l where p_l is the lth prime number in the sequence of natural numbers does not exist either.

235. It can easily be seen that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$ (check it!). Therefore it is sufficient to show that the expression

$$P(a, b, c)=(a+b+c)^{333}-a^{333}-b^{333}-c^{333}$$

is divisible by $a+b$, by $b+c$ and by $a+c$. It is evident that the expression $P(x, b, c)=(x+b+c)^{333}-x^{333}-b^{333}-c^{333}$ regarded as a polynomial in the variable x turns into zero when $-b$ is substituted for x; therefore $P(x, b, c)$ is divisible by $x-(-b)=x+b$, and consequently $P(a, b, c)$ is divisible by $a+b$. It can similarly be proved that $P(a, b, c)$ is divisible by both $b+c$ and $c+a$ (this also follows from the fact that the letters a, b and c are involved symmetrically in the expression $P(a, b, c)$).

236. We have

$$a^{10}+a^5+1=\frac{(a^5)^3-1}{a^5-1}=\frac{a^{15}-1}{a^5-1}=$$
$$=\frac{(a^3)^5-1}{(a-1)(a^4+a^3+a^2+a+1)}=\frac{(a^3-1)(a^{12}+a^9+a^6+a^3+1)}{(a-1)(a^4+a^3+a^2+a+1)}=$$
$$=\frac{(a^2+a+1)(a^{12}+a^9+a^6+a^3+1)}{a^4+a^3+a^2+a+1}$$

The division shows that

$$\frac{a^{12}+a^9+a^6+a^3+1}{a^4+a^3+a^2+a+1}=a^8-a^7+a^5-a^4+a^3-a+1$$

and consequently

$$a^{10}+a^5+1=(a^2+a+1)(a^8-a^7+a^5-a^4+a^3-a+1)$$

237. *First solution.* Let us denote the given polynomials as B and A respectively. Then we can write

$$\begin{aligned}B-A&=(x^{9999}-x^9)+(x^{8888}-x^8)+(x^{7777}-x^7)+\\&+(x^{6666}-x^6)+(x^{5555}-x^5)+(x^{4444}-x^4)+\\&+(x^{3333}-x^3)+(x^{2222}-x^2)+(x^{1111}-x)=\\&=x^9[(x^{10})^{999}-1]+x^8[(x^{10})^{888}-1]+x^7[(x^{10})^{777}-1]+\\&+x^6[(x^{10})^{666}-1]+x^5[(x^{10})^{555}-1]+x^4(x^{10})^{444}-1]+\\&+x^3[(x^{10})^{333}-1]+x^2[(x^{10})^{222}-1]+[(x^{10})^{111}-1]\end{aligned}$$

Here every expression in the parentheses is divisible by $x^{10}-1$ and, consequently, by $A=(x^{10}-1)/(x-1)$ as well. We thus see that $B-A$ is divisible by A, and therefore B is divisible by A.

Second solution. We have

$$\begin{aligned}x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1&=\frac{x^{10}-1}{x-1}=\\&=\frac{(x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ \dots\ (x-\alpha_9)}{x-1}=\\&=(x-\alpha_1)(x-\alpha_2)\ \dots\ (x-\alpha_9)\end{aligned}$$

where $\alpha_k=\cos 2k\pi/10+i\sin 2k\pi/10$ $(k=1, 2, \dots, 9)$ because the roots of the equation $x^{10}-1=0$ (that is the tenth roots of unity) are expressed in just this way. Consequently, to prove the required assertion it suffices to check that the expression

$$x^{9999}+x^{8888}+x^{7777}+x^{6666}+x^{5555}+x^{4444}+x^{3333}+x^{2222}+x^{1111}+1$$

is divisible by each of the binomials $(x-\alpha_1)$, $(x-\alpha_2)$, ..., $(x-\alpha_9)$. But this divisibility is equivalent to the fact that the equation

$$\begin{aligned}x^{9999}+x^{8888}+x^{7777}+x^{6666}+x^{5555}+x^{4444}+\\+x^{3333}+x^{2222}+x^{1111}+1=0\quad(*)\end{aligned}$$

has the roots equal to $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_9$. Let us verify that these values of x do in fact satisfy equation (*); indeed, since $\alpha_k^{10}=1$ $(k=1, 2, 3, \dots, 9)$ we have

$$\alpha_k^{9999}=\alpha_k^{9990+9}=\left(\alpha_k^{10}\right)^{999}\alpha_k^9=\alpha_k^9$$

$$\alpha_k^{8888}=\alpha_k^{8880+8}=\left(\alpha_k^{10}\right)^{888}\alpha_k^8=\alpha_k^8\quad\text{etc.}$$

and

$$\alpha_k^{9999} + \alpha_k^{8888} + \alpha_k^{7777} + \alpha_k^{6666} + \alpha_k^{5555} + \alpha_k^{4444} + \alpha_k^{3333} + \alpha_k^{2222} + \alpha_k^{1111} + 1 =$$
$$= \alpha_k^9 + \alpha_k^8 + \alpha_k^7 + \alpha_k^6 + \alpha_k^5 + \alpha_k^4 + \alpha_k^3 + \alpha_k^2 + \alpha_k + 1 =$$
$$= 0 \quad (k = 1, 2, \ldots, 9)$$

238. *First solution.* We have

$$a^3 + b^3 + c^3 - 3abc =$$
$$= a^3 + 3ab(a+b) + b^3 + c^3 - 3abc - 3ab(a+b) =$$
$$= a^3 + 3a^2b + 3ab^2 + b^3 + c^3 - 3ab(c+a+b) =$$
$$= (a+b)^3 + c^3 - 3ab(a+b+c) =$$
$$= [(a+b)+c][(a+b)^2 - (a+b)c + c^2] - 3ab(a+b+c) =$$
$$= (a+b+c)[(a+b)^2 - (a+b)c + c^2 - 3ab] =$$
$$= (a+b+c)(a^2 + 2ab + b^2 - ac - bc + c^2 - 3ab) =$$
$$= (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$$

Second solution. Let us replace the letter a by x and put $x + b + c = 0$. On transforming the expression $(x+b+c)^3$ we conclude that $x^3 + b^3 + c^3 - 3xbc = 0$ when $x + b + c = 0$. Therefore the value $x = -b - c$ is a root of the equation $x^3 - 3bcx + b^3 + c^3 = 0$, and consequently the expression $a^3 + b^3 + c^3 - 3abc$ is divisible by $a + b + c$. On performing the division (to this end it is convenient to regard the expressions $a^3 - 3abc + b^3 + c^3$ and $a + b + c$ as being arranged in ascending powers of the variable a) we arrive at the former result:

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$$

(b) Let us choose two numbers a and b such that the equality

$$x^3 + px + q = x^3 + a^3 + b^3 - 3abx$$

is fulfilled. To this end it is sufficient that a and b should satisfy the relations $a^3 + b^3 = q$ and $ab = -p/3$. These two relations are a system of two equations in the two unknowns a and b from which a and b can be found. We have $a^3 + b^3 = q$ and $a^3b^3 = -p^3/27$; it follows that a^3 and b^3 are the roots of the quadratic equation $z^2 - qz - p^3/27 = 0$, and consequently *

$$a = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad b = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{2} + \frac{p^3}{27}}} \qquad (*)$$

* The numbers a and b defined by formulas (*) are real when $q^2/4 + p^3/27 \geqslant 0$. If $q^2/4 + p^3/27 < 0$ we have the third roots of complex numbers in formulas (*). In this case the numbers a and b are also complex; they can be found using the formula for the nth root of a complex number. On applying

Now, by virtue of the result established in Problem 238 (a), we obtain

$$x^3+px+q=x^3+a^3+b^3-3abx=$$
$$=(a+b+x)(a^2+b^2+x^2-ab-ax-bx)$$

Consequently, the solution of the given cubic equation reduces to the solution of the first-degree equation

$$a+b+x=0$$

and the quadratic equation

$$x^2-(a+b)x+a^2+b^2-ab=0$$

From the first equation we find

$$x_1=-a-b$$

that is

$$x_1=-\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

It follows that

$$x_2=\frac{a+b}{2}+\frac{(a-b)\sqrt{3}}{2}i, \quad x_3=\frac{a+b}{2}-\frac{(a-b)\sqrt{3}}{2}i$$

where a and b are determined by formulas (*).

239. *First solution.* Let us denote $\sqrt{a+x}$ by y; then we obtain the following system of two equations:

$$\sqrt{a+x}=y, \quad \sqrt{a-y}=x$$

On squaring these equations we find

$$a+x=y^2, \quad a-y=x^2$$

Let us subtract the second of the last relations from the first one; this results in

$$x+y=y^2-x^2$$

The last equality can be rewritten as

$$x^2-y^2+x+y=(x+y)(x-y+1)=0$$

Now we see that there can be the following two possibilities:
(1) $x+y=0$, $y=-x$ and $x^2-x-a=0$ whence

$$x_{1,2}=\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$$

this formula we can take as a any of the three values of the third root of the complex number $q/2+\sqrt{q^2/4+p^3/27}$, after which b can be found from the relation $ab=-p/3$.

(2) $x-y+1=0$; then $y=x+1$ and $x^2+x+1-a=0$ whence

$$x_{3,4}=-\frac{1}{2}\pm\sqrt{a-\frac{3}{4}}$$

It can be verified directly that the expressions x_1, x_2, x_3 and x_4 thus determined are in fact the roots of the given equation provided that the signs of the radicals occurring in this equation are chosen appropriately *.

Second solution. Let us eliminate the radical in the given equation:

$$a-\sqrt{a+x}=x^2$$

$$(a-x^2)^2=a+x$$

and, finally,

$$x^4-2ax^2-x+a^2-a=0$$

Thus, we have arrived at a fourth-degree equation in x; with respect to a it is a quadratic equation. Let us find a from it. To this end we regard temporarily x as a given quantity and express a in terms of x:

$$a^2-(2x^2+1)a+x^4-x=0$$

$$a=\frac{2x^2+1\pm\sqrt{4x^4+4x^2+1-4x^4+4x}}{2}=$$

$$=\frac{2x^2+1\pm\sqrt{4x^2+4x+1}}{2}=\frac{2x^2+1\pm(2x+1)}{2}$$

and, finally,

$$a_1=x^2+x+1,\quad a_2=x^2-x$$

We see that the equation

$$a^2-(2x^2+1)a+x^4-x=0$$

possesses the roots

$$a_1=x^2+x+1,\quad a_2=x^2-x$$

whence, by virtue of the general properties of the roots of a quadratic equation, we conclude that

$$a^2-(2x^2+1)a+x^4-x=(a-a_1)(a-a_2)=$$

$$=(a-x^2-x-1)(a-x^2+x)$$

Hence, the given equation takes the form

$$(x^2-x-a)(x^2+x-a+1)=0$$

* It should be noted that if all the radicals are considered positive then the equation possesses only one root $x_3=-1/2+\sqrt{a-3/4}$ in case $a\geqslant 1$ and has no roots at all in case $a<1$.

and can easily be solved:

$$x_{1,2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+a}=\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$$

$$x_{3,4}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+a-1}=-\frac{1}{2}\pm\sqrt{a-\frac{3}{4}}$$

240. *First solution.* Let us denote

$$x^2+2ax+\frac{1}{16}=y, \quad -a+\sqrt{a^2+x-\frac{1}{16}}=y_1$$

The given equation takes the form

$$y=y_1$$

Now let us express x in terms of y_1. Simple calculations result in

$$x=y_1^2+2ay_1+\frac{1}{16}$$

We thus see that x is expressed in terms of y_1 in just the same way as y is expressed in terms of x. It follows that the graphs of the functions

$$y=x^2+2ax+\frac{1}{16}$$

and

$$y_1=-a+\sqrt{a^2+x-\frac{1}{16}}$$

are parabolas located symmetrically about the bisector of the first quadrant (see Fig. 27; to every point $x=x_0$, $y=y_0$ lying on the first graph there corresponds the point $x=y_0$, $y=x_0$ lying on the second graph which is symmetric to the former point about the bisector of the first quadrant). The points of intersection of the two graphs correspond to those values of x for which $y=y_1$, that is to the roots of the given equation. These points must necessarily lie on the axis of symmetry of both curves, that is they satisfy the condition

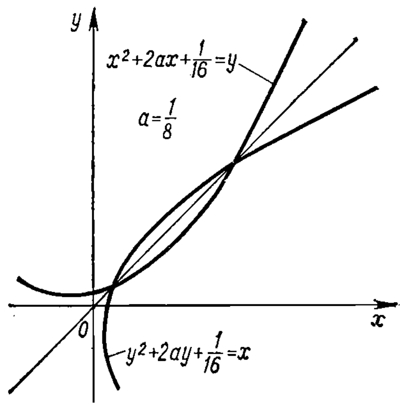


Fig. 27

$$y=x=y_1$$

On solving the equation $y=x$ which can be written as

$$x^2+2ax+\frac{1}{16}=x$$

we obtain

$$x_{1,2}=\frac{1-2a}{2}\pm\sqrt{\left(\frac{1-2a}{2}\right)^2-\frac{1}{16}}$$

It can readily be verified that for $0<a<1/4$ both roots are real and do in fact satisfy the original equation.

Second solution. This problem can also be solved in a more traditional manner without using graphs. On eliminating the radical in the given equation we obtain

$$\left(x^2+2ax+a+\frac{1}{16}\right)^2=a^2+x-\frac{1}{16}$$

Next we open parentheses and collect like terms, which yields

$$x^4+4ax^3+\left(4a^2+2a+\frac{1}{8}\right)x^2+\left(4a^2+\frac{1}{4}a-1\right)x+$$
$$+\frac{a}{8}+\frac{1}{16}+\frac{1}{16^2}=0$$

The left-hand member of the resultant equation can be factored as

$$\left[x^4+(2a-1)x^3+\frac{1}{16}x^2\right]+\left[(2a+1)x^3+(4a^2-1)x^2+\left(\frac{a}{8}+\frac{1}{16}\right)x\right]+$$
$$+\left[\left(2a+\frac{17}{16}\right)x^2+\left(4a^2+\frac{a}{8}-\frac{17}{16}\right)x+\left(\frac{a}{8}+\frac{1}{16}+\frac{1}{16^2}\right)\right]=$$
$$=\left[x^2+(2a-1)x+\frac{1}{16}\right]\left[x^2+(2a+1)x+\left(2a+\frac{17}{16}\right)\right]$$

Now we readily obtain the solutions:

$$x^2+(2a-1)x+\frac{1}{16}=0$$

whence

$$x_{1,2}=\frac{1-2a}{2}\pm\sqrt{\left(\frac{1-2a}{2}\right)^2-\frac{1}{16}}$$

and

$$x^2+(2a+1)x+2a+\frac{17}{16}=0$$

whence

$$x_{3,4}=-\frac{1+2a}{2}\pm\sqrt{\left(\frac{1+2a}{2}\right)^2-2a-\frac{17}{16}}$$

For $0<a<1/4$ the first two roots are real and satisfy the original equation; the last two roots are complex.

241. For the left-hand member of the given equation to be real when x is real it is necessary that all the radicands should be positive. On denoting these positive radicands beginning with the last one (which is equal to $3x$) up to the first one as $y_1^2, y_2^2, y_3^2, \ldots$

$\ldots, y_{n-1}^2, y_n^2$ respectively we can write

$$3x = x + 2x = y_1^2$$
$$x + 2y_1 = y_2^2$$
$$x + 2y_2 = y_3^2$$
$$\cdots\cdots$$
$$x + 2y_{n-2} = y_{n-1}^2$$
$$x + 2y_{n-1} = y_n^2$$

where all the numbers $y_1, y_2, \ldots, y_n$ are real and positive. The original equation itself takes the form

$$y_n = x$$

Let us prove that $y_1 = x$. Indeed, let us suppose, for definiteness, that $x > y_1$. Then the comparison of the first and the second of the above equalities shows that $y_1 > y_2$. Similarly, from the second and the third equalities we find that $y_2 > y_3$; further, we analogously obtain the inequalities

$$y_3 > y_4 > \ldots > y_{n-1} > y_n$$

Thus, for $x > y_1$ we have $x > y_n$, which contradicts the equation $y_n = x$. It can similarly be shown that for $x < y_1$ the equality $y_n = x$ cannot be fulfilled either (in this case the inequality $x < y_n$ must necessarily hold).

Since $y_1^2 = 3x$, it follows that the relation

$$3x = x^2$$

must hold, whence we readily conclude that only the following two values of x are admissible:

$$x_1 = 3, \quad x_2 = 0$$

The direct verification shows that both these values satisfy the given equation.

Remark. We shall also mention one more method for the solution of the equation

$$\underbrace{\sqrt{x + 2\sqrt{x + 2\sqrt{x + \ldots + 2\sqrt{x + 2x}}}}}_{n \text{ radical signs}} = x \qquad (*)$$

On replacing the last letter x on the left-hand side of equation (*) by the value of x expressed by (*) we obtain

$$x = \underbrace{\sqrt{x + 2\sqrt{x + 2\sqrt{x + \ldots + 2\sqrt{x + 2x}}}}}_{2n \text{ radical signs}}$$

Further, let us replace the last letter x by the same expression; again and again this yields

$$x=\sqrt{\underbrace{x+2\sqrt{x+2\sqrt{x+\ \dots\ +2\sqrt{x+2x}}}}_{3n\text{ radical signs}}}=$$

$$=\sqrt{\underbrace{x+2\sqrt{x+2\sqrt{x+\ \dots\ +2\sqrt{x+2x}}}}_{4n\text{ radical signs}}}=\dots$$

On the basis of these relations we can write

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ \dots}}}=$$

$$=\lim_{N\to\infty}\sqrt{\underbrace{x+2\sqrt{x+2\sqrt{x+\ \dots\ +2\sqrt{x+2x}}}}_{N\text{ radical signs}}}\qquad (**)$$

whence it follows that

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ \dots}}}=$$

$$=\sqrt{x+2\left[\sqrt{x+2\sqrt{x+2\sqrt{x+\ \dots}}}\right]}=\sqrt{x+2x}\qquad (***)$$

From the last relation we find $x=\sqrt{3x}$, that is $x^2=3x$, and consequently $x_1=0$ and $x_2=3$. In particular, this method of solution readily shows that the roots of equation (*) are independent of n because equation (**) does not involve n).

This argument cannot be, of course, regarded as a rigorous solution of the problem since we have not proved the existence of limit (**) and the validity of transformation (***). It should be noted however that it is in fact possible to modify this argument to elaborate a rigorous solution.

242. Let us consecutively simplify the fraction on the right-hand side:

$$1+\frac{1}{x}=\frac{x+1}{x};\quad 1+\frac{1}{\frac{x+1}{x}}=1+\frac{x}{x+1}=\frac{2x+1}{x+1};$$

$$1+\frac{1}{\frac{2x+1}{x+1}}=1+\frac{x+1}{2x+1}=\frac{3x+2}{2x+1};\ \dots.$$

Finally we arrive at an equation of the form

$$\frac{ax+b}{cx+d}=x$$

where a, b, c and d are some integers yet unknown. This equation is equivalent to the quadratic equation $x(cx+d)=ax+b$. It follows that the original equation possesses *not more than two* different solutions (this equation cannot turn into an identity because, if otherwise, any value of x would satisfy it, which is false since $x=0$ obviously does not satisfy the equation).

These two roots of the equation can easily be found. Indeed, let us suppose that x is such that

$$1+\frac{1}{x}=x$$

Then, simplifying consecutively the given fraction "beginning with its end", we obtain

$$1+\frac{1}{x}=x; \quad 1+\frac{1}{x}=x; \quad 1+\frac{1}{x}=x; \ \ldots.$$

and finally arrive at the identity

$$x=x$$

Thus, we see that the roots of the equation $1+\frac{1}{x}=x$ (it is equivalent to the quadratic equation $x^2-x-1=0$) which are equal to

$$x_1=\frac{1+\sqrt{5}}{2}, \quad x_2=\frac{1-\sqrt{5}}{2}$$

satisfy the given equation, and this equation has no other roots.

Remark. We shall also mention another method for the solution of the problem (cf. the remark to Problem 241). Let us replace x on the left-hand side of the given equation by the expression of x in the form of a *terminating continued* fraction given by the equation itself. This results in an equation of the same form which however involves $2n$ fraction lines. Continuing this process we consecutively obtain fractions with an increasing number of fraction lines. On the basis of this transformation we can write

$$x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}} = 1+\lim_{N\to\infty}\underbrace{\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots+\cfrac{1}{1+\cfrac{1}{1}}}}}}_{\text{fraction line is repeated N times}} \qquad (*)$$

where on the left-hand side there is a *nonterminating continued fraction* involving infinitely many fraction lines. The last expression implies

$$x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}} = 1+\cfrac{1}{\left[1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}\right]} \qquad (**)$$

that is we arrive at the quadratic equation for x found in the former solution of the problem. The latter solution of the original equation shows directly that its roots cannot depend on n.

This argument cannot be regarded as a rigorous solution of the problem because we have not proved the existence of limit (*), the equality between x and this limit and the validity of transformation (**). However, it should be noted that this argument can be modified to obtain a quite rigorous solution.

243. We have

$$x+3-4\sqrt{x-1}=x-1-4\sqrt{x-1}+4=$$
$$=(\sqrt{x-1})^2-4\sqrt{x-1}+4=(\sqrt{x-1}-2)^2$$

and, similarly,

$$x+8-6\sqrt{x-1}=x-1-6\sqrt{x-1}+9=(\sqrt{x-1}-3)^2$$

Hence, the given equation can be rewritten in the form

$$\sqrt{(\sqrt{x-1}-2)^2}+\sqrt{(\sqrt{x-1}-3)^2}=1$$

Since all the roots are considered positive the equation can also be written as

$$|\sqrt{x-1}-2|+|\sqrt{x-1}-3|=1$$

where $|y|$ designates the absolute value of the number y.

Now let us consider separately the following possible cases.

1°. If $\sqrt{x-1}-2\geqslant 0$ and $\sqrt{x-1}-3\geqslant 0$, that is if $\sqrt{x-1}\geqslant 3$, then we have $x-1\geqslant 9$ whence $x\geqslant 10$. In this case $|\sqrt{x-1}-2|=\sqrt{x-1}-2$ and $|\sqrt{x-1}-3|=\sqrt{x-1}-3$; therefore the equation takes the form

$$\sqrt{x-1}-2+\sqrt{x-1}-3=1$$

whence

$$2\sqrt{x-1}=6,\quad x-1=9,\quad x=10$$

2°. If $\sqrt{x-1}-2\geqslant 0$ and $\sqrt{x-1}-3\leqslant 0$, that is if $\sqrt{x-1}\geqslant 2$, $x\geqslant 5$ but $\sqrt{x-1}\leqslant 3$, $x\leqslant 10$, then $|\sqrt{x-1}-2|=$ $=\sqrt{x-1}-2$, $|\sqrt{x-1}-3|=-\sqrt{x-1}+3$ and the equation turns into the identity

$$\sqrt{x-1}-2-\sqrt{x-1}+3=1$$

This means that *all* the values of x lying between $x=5$ and $x=10$ satisfy the given equation.

3°. If $\sqrt{x-1}-2\leqslant 0$ and $\sqrt{x-1}-3\leqslant 0$, that is if $\sqrt{x-1}\leqslant 2$, then $x\leqslant 5$; in this case we have $|\sqrt{x-1}-2|=$ $=-\sqrt{x-1}+2$, $|\sqrt{x-1}-3|=-\sqrt{x-1}+3$, and the equation takes the form

$$-\sqrt{x-1}+2-\sqrt{x-1}+3=1$$

whence

$$2\sqrt{x-1}=4, \quad x-1=4, \quad x=5$$

4°. The case when $\sqrt{x-1}-2 \leqslant 0$ and $\sqrt{x-1}-3 \geqslant 0$ is obviously impossible.

Thus, the solutions of the equation are *all* the values of x lying between $x=5$ and $x=10$: $5 \leqslant x \leqslant 10$.

244. To solve the given equation we shall first determine its roots lying within the interval from 2 to ∞ and then, consecutively, the roots lying within the intervals from 1 to 2, from 0 to 1, from -1 to 0 and from $-\infty$ to -1.

1°. Let $x \geqslant 2$. Then we have $x+1>0$, $x>0$, $x-1>0$ and $x-2 \geqslant 0$; therefore $|x+1|=x+1$, $|x|=x$, $|x-1|=x-1$ and $|x-2|=x-2$. We thus arrive at the equation

$$x+1-x+3(x-1)-2(x-2)=x+2$$

which is *satisfied identically*.

Hence, any number greater than or equal to 2 is a root of the given equation.

2°. Let $1 \leqslant x < 2$. Then $x+1>0$, $x>0$, $x-1 \geqslant 0$ and $x-2<0$; consequently $|x+1|=x+1$, $|x|=x$, $|x-1|=x-1$ and $|x-2|=-(x-2)$.

Thus, we obtain the equation

$$x+1-x+3(x-1)+2(x-2)=x+2$$

From this equation we find $4x=8$, whence $x=2$. Since the number $x=2$ does not belong to the interval $1 \leqslant x < 2$, the given equation possesses no roots which are greater than or equal to 1 and are smaller than 2.

3°. Let $0 \leqslant x < 1$. Then we have

$$|x+1|=x+1, \quad |x|=x$$

and

$$|x-1|=-(x-1), \quad |x-2|=-(x-2)$$

Hence,

$$x+1-x-3(x-1)+2(x-2)=x+2, \quad x=-1$$

Since the value $x=1$ lies outside the interval $0 \leqslant x < 1$, there are no roots which are greater than or equal to 0 and are less than 1.

4°. Let $-1 \leqslant x < 0$. Then $|x+1|=x+1$. In this case

$$|x|=-x, \quad |x-1|=-(x-1), \quad |x-2|=-(x-2)$$

and

$$x+1+x-3(x-1)+2(x-2)=x+2$$

which is impossible, that is the interval $-1 \leqslant x < 0$ does not contain roots either.

5°. Let $x < -1$. Then $|x+1| = -(x+1)$, $|x| = -x$, $|x-1| = -(x-1)$ and $|x-2| = -(x-2)$

We obtain

$$-(x+1)+x-3(x-1)+2(x-2)=x+2,\ x=-2$$

Hence, there is one more root $x = -2$.

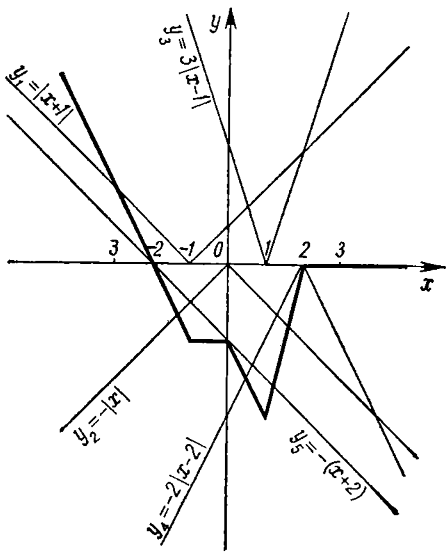


Fig. 28

Finally, we conclude that the roots of the equation are the number -2 and all the numbers greater than or equal to 2.

Remark. The result of the present problem becomes particularly clear if we construct the graph of the function

$$y = |x+1| - |x| + 3|x-1| - 2|x-2| - (x+2)$$

In Fig. 28 the thin lines represent the graphs of the functions $y_1 = |x+1|$, $y_2 = -|x|$, $y_3 = 3|x-1|$, $y_4 = -2|x-2|$ and $y_5 = -(x+2)$, and the heavy line the graph of the function $y = y_1 + y_2 + y_3 + y_4 + y_5$ (here we have performed the "addition" of the graphs). As is readily seen from the figure, the variable y turns into zero on the ray $x \geqslant 2$ and at the separate point $x = -2$.

245. Let us denote the right-hand side of the given equation of the nth degree as $f_n(x)$. It is easily seen that $f_1(x) = 0$, that is the equation $1 - x = 0$ has the root $x_1 = 1$; the equation $f_2(x) = 0$ has the form $x(x-1) - 2x + 2 = 0$ or, equivalently, $x^2 - 3x + 2 = 0$; this equation has the roots $x_1 = 1$ and $x_2 = 2$. Now let us prove that the *equation* $f_n(x) = 0$ *possesses the following roots*:

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \ \ldots, \ x_{n-1} = n-1, \quad x_n = n \qquad (*)$$

We shall make use of the *method of mathematical induction*. To this end we assume that the assertion has already been proved for the equation $f_n(x) = 0$ and then show that under this assumption the equation $f_{n+1}(x) = 0$ possesses the same roots (*) and an additional root $x_{n+1} = n+1$. First of all, since

$$f_{n+1}(x) = f_n(x) + (-1)^{n+1} \frac{x(x-1)(x-2)\ldots(x-n+1)(x-n)}{(n+1)!}$$

it is clear that if the equation $f_n(x) = 0$ has roots (*), then the same roots also satisfy the equation $f_{n+1}(x) = 0$. Finally, the equality $f_{n+1}(n+1) = 0$ can be written in the form

$$1 - \frac{n+1}{1} + \frac{(n+1)n}{1\cdot 2} - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \ldots$$
$$\ldots + (-1)^{n+1} \frac{(n+1)n(n-1)\ldots 2\cdot 1}{(n+1)!} = 0$$

that is

$$1 - C(n+1, 1) + C(n+1, 2) - C(n+1, 3) + \ldots$$
$$\ldots + (-1)^{n+1} C(n+1, n+1) = 0 \qquad (**)$$

where $C(n+1, k) = \frac{(n+1)n(n-1)\ldots(n-k+2)}{k!}$ are the so-called *binomial coefficients*. By Newton's binomial formula, the right-hand side of (**) is equal to $(1-1)^{n+1} = 0$, whence it follows that the number $x_{n+1} = n+1$ is also a root of the equation $f_{n+1}(x) = 0$.

246. Let us denote by $\{x\}$ the *fractional part* of the number x: $\{x\} = x - [x]$ (see page 37). It is evident that $0 \leqslant \{x\} < 1$ and $[x] = x - \{x\}$. Thus, the given equation takes the form

$$x^3 - (x - \{x\}) = 3, \quad \text{that is} \quad x^3 - x = 3 - \{x\}$$

whence it follows that $2 < x^3 - x \leqslant 3$. Further, for $x \geqslant 2$ we have $x^3 - x = x(x^2 - 1) \geqslant 2(4-1) = 6 > 3$; for $x < -1$ we have $x^2 - 1 > 0$ and $x^3 - x = x(x^2-1) < 0 < 2$; for $x = -1$ we have $x^3 - x = 0 < 2$; for $-1 < x \leqslant 0$ we have $x^3 - x \leqslant -x < 1$ and for $0 < x \leqslant 1$ we have $x^3 - x < x^3 \leqslant 1$. Therefore there must be $1 < x < 2$, and consequently $[x] = 1$. Now the original equation can be written in the form

$$x^3 - 1 = 3 \quad \text{whence} \quad x^3 = 4, \quad \text{that is} \quad x = \sqrt[3]{4}$$

Thus, $x = \sqrt[3]{4}$ is nothing other than the (single) solution to the problem.

247. From the first equation of the given system we immediately obtain

$$y^2 = x^2, \quad y = \pm x$$

The substitution of this expression of y^2 into the second equation yields

$$(x - a)^2 + x^2 = 1 \qquad (*)$$

This is a quadratic equation; in the general case it determines two values of x. Since to every value of x there correspond two values of y, the total number of the solutions of the problem is equal to *four.*

The number of the solutions of the system reduces to *three* when one of the values of x is equal to zero; to the value $x = 0$ (and only to this value) there corresponds a single solution $y = 0$ and not two different values $y = \pm x$. On substituting $x = 0$ into equation (*) we find

$$a^2 = 1 \text{ whence } a = \pm 1$$

For only these values of a the system possesses three solutions.

The number of the solutions of the system reduces to *two* when the equation for x has only one solution. For the quadratic equation $(x - a)^2 + x^2 = 1$ which can be written as $2x^2 - 2ax + a^2 - 1 = 0$ to have only one solution (in this case the two roots coincide) there must be

$$a^2 - 2(a^2 - 1) = 0 \text{ whence } a^2 = 2, \text{ that is } a = \pm\sqrt{2}$$

For these values of a the system possesses two solutions.

248. (a) On solving the system we find

$$x = \frac{a^3 - 1}{a^2 - 1}, \quad y = \frac{-a^2 + a}{a^2 - 1}$$

It follows that if $a + 1 \neq 0$ and $a - 1 \neq 0$ then the system has only one solution $x = (a^2 + a + 1)/(a + 1)$, $y = -a/(a + 1)$. If $a = -1$ or $a = +1$ the formulas we have derived do not make sense. In the case $a = -1$ we arrive at the system

$$\left.\begin{aligned} -x + y &= 1 \\ x - y &= 1 \end{aligned}\right\}$$

which is inconsistent (that is it has no solutions at all) and in the case $a = +1$ we obtain the system

$$\left.\begin{aligned} x + y &= 1 \\ x + y &= 1 \end{aligned}\right\}$$

possessing infinitely many solutions (in this case x is arbitrary and y is expressed by the formula $y = 1 - x$).

(b) On solving the system we obtain

$$x = \frac{a^4 - 1}{a^2 - 1}, \quad y = \frac{-a^3 + a}{a^2 - 1}$$

Thus, in this case as well the system has only one solution $x = a^2 + 1$, $y = -a$ when $a^2 - 1 \neq 0$. As to the cases when $a = -1$ or $a = 1$, we arrive at the systems

$$\left.\begin{aligned} -x + y &= -1 \\ x - y &= 1 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x + y &= 1 \\ x + y &= 1 \end{aligned}\right\}$$

respectively each of which possesses infinitely many solutions.

(c) From the first and the second equations we find

$$y + z = 1 - ax \quad \text{and} \quad ay + z = a - x$$

These two relations can be regarded as a system of two equations in the two unknowns y and z; on solving the system we obtain

$$y = \frac{a - x - 1 + ax}{a - 1} = \frac{(a-1)(1+x)}{a-1}$$

$$z = \frac{a(1 - ax) - a + x}{a - 1} = \frac{-x(a^2 - 1)}{a - 1}$$

Thus, if $a \neq 1$ then $y = 1 + x$, $z = -(1 + a)x$; the substitution of these values of y and z into the third equation results in

$$x + (1 + x) - a(1 + a)x = a^2, \quad x(2 - a - a^2) =$$
$$= a^2 - 1, \quad -x(a + 2)(a - 1) = a^2 - 1$$

Therefore for $a - 1 \neq 0$ and $a + 2 \neq 0$ the system has a single solution:

$$x = -\frac{a^2 - 1}{(a + 2)(a - 1)} = -\frac{a + 1}{a + 2},$$

$$y = 1 + x = \frac{1}{a + 2}, \quad z = -(a + 1)x = \frac{(a + 1)^2}{a + 2}$$

In the cases when $a = 1$ or $a = -2$ we arrive at the systems

$$\left.\begin{aligned} x + y + z &= 1 \\ x + y + z &= 1 \\ x + y + z &= 1 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} -2x + y + z &= 1 \\ x - 2y + z &= -2 \\ x + y - 2z &= 4 \end{aligned}\right\}$$

respectively; the first of them has infinitely many solutions whereas the other has no solutions at all (the addition of the first two equations of the second system results in the equation $-x - y + 2z = -1$ which contradicts the third equation).

249. Let us subtract the second equation from the first one and the sixth equation from the fifth one; on equating the two expressions for $x_2 - x_3$ thus obtained we find

$$\alpha_1(\alpha_2-\alpha_3)=\alpha_4(\alpha_2-\alpha_3) \quad \text{whence} \quad (\alpha_1-\alpha_4)(\alpha_2-\alpha_3)=0$$

In just the same way, on finding two different expressions for each of the differences $x_1 - x_2$ and $x_1 - x_3$, we arrive at two more relations:

$$(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)=0 \quad \text{and} \quad (\alpha_1-\alpha_3)(\alpha_2-\alpha_4)=0$$

From the first of the three relations we have derived it follows that $\alpha_1 = \alpha_4$ or $\alpha_2 = \alpha_3$. For definiteness, let us assume that $\alpha_2 = \alpha_3 = \alpha$. From the second relation it follows that $\alpha_1 = \alpha$ or $\alpha_4 = \alpha$. For each of these equalities the third relation turns into an identity. Thus, *for the given system to be consistent it is necessary that three of the four numbers* α_1, α_2, α_3 *and* α_4 *should be equal to one another.*

Now let us suppose that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, $\alpha_4 = \beta$. Using the expressions for the differences $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$ which we derived earlier and with the aid of which the relationships between α_1, α_2, α_3 and α_4 were established we immediately obtain

$$x_1 = x_2 = x_3$$

Now let us denote $x_1 = x_2 = x_3$ by x and x_4 by y. Then the given system of six equations in four unknowns goes into two equations in two unknowns:

$$2x = \alpha^2, \quad x + y = \alpha\beta$$

From the last system we find

$$x = \frac{\alpha^2}{2}, \quad y = \alpha\left(\beta - \frac{\alpha}{2}\right)$$

Remark. Using an analogous argument we can show that a more general system of the form

$$\left.\begin{array}{r} x_1 + x_2 + \dots + x_{m-1} + x_m = \alpha_1\alpha_2 \dots \alpha_{m-1}\alpha_m \\ x_1 + x_2 + \dots + x_{m-1} + x_{m+1} = \alpha_1\alpha_2 \dots \alpha_{m-1}\alpha_{m+1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{n+m-1} + x_{n+m-2} + \dots + x_{n-1} + x_n = \alpha_{n+m-1}\alpha_{n+m-2} \dots \alpha_{n-1}\alpha_n \end{array}\right\}$$

consisting of $C(n, m)$ equations in n unknowns ($n > m + 1$) is solvable only in the following two cases:

1°. $\alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = \alpha$, $\alpha_n = \beta$ (in this case $x_1 = x_2 = \dots = x_{n-1} = \alpha^n/n$, $x_n = \alpha^{n-1}\left(\beta - \frac{n-1}{n}\alpha\right)\big)$.

2°. In the sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ there are $n - m + 1$ (or more) numbers equal to zero (in this case $x_1 = x_2 = \dots = x_n = 0$).

250. From the first equation we obtain $x = 2 - y$; the substitution of this expression into the second equation results in the relation

$$2y - y^2 - z^2 = 1$$

which can be rewritten as

$$z^2 + y^2 - 2y + 1 = 0, \text{ whence } z^2 + (y-1)^2 = 0$$

Each of the two summands on the left-hand side of the last equality is nonnegative and, consequently, it must be equal to zero. It follows that

$$z = 0, \quad y = 1$$

and hence

$$x = 1$$

Thus, the system possesses a *single* real solution.

251. If $x^4 + y^4 = 1$ then $x^4 = 1$, $y^4 = 0$ or $x^4 = 0$, $y^4 = 1$ or, finally, $0 < x^4$, $y^4 < 1$ (because the numbers x^4 and y^4 are nonnegative). For $x^4 < 1$ and $y^4 < 1$ we also have $|x| < 1$ and $|y| < 1$ whence

$$x^4 = |x^4| = |x^3| \cdot |x| < |x^3|, \quad y^4 < |y^3|$$
$$\text{and } |x^3| + |y^3| > x^4 + y^4 = 1$$

It follows that two numbers x and y of one sign satisfying the conditions $|x| < 1$ and $|y| < 1$ cannot serve as a solution to the given system. It is even more evident that two numbers x and y of opposite signs such that $|x| < 1$ and $|y| < 1$ cannot serve as a solution either because in this case

$$x^3 + y^3 \leqslant |x^3 + y^3| < \max(|x^3|, \quad |y^3|) < 1$$

Thus, the solutions of the system can only involve values of x and y such that $x^4 = 1$ and $y^4 = 0$ or $x^4 = 0$ and $y^4 = 1$, that is $x = \pm 1$ and $y = 0$ or $x = 0$ and $y = \pm 1$. It is clear that among these four pairs of values of x and y only the *two* pairs $x = 1$, $y = 0$ and $x = 0$, $y = 1$ are the solutions of the system.

252. One solution (or, more precisely, a system of solutions) is quite evident: $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ and x is arbitrary; therefore in what follows we shall assume that at least one of the numbers x_i ($i = 1, 2, 3, 4, 5$) is different from zero. Further, from the first and the last equations of the system we derive

$$x_3 = xx_2 - x_1 \text{ and } x_5 = xx_1 - x_2 \qquad (*)$$

Similarly, the second and the last but one equations yield

$$x_4 = xx_3 - x_2 \text{ and } x_4 = xx_5 - x_1 \qquad (**)$$

On substituting into (**) the values of x_3 and x_5 expressed by formulas (*) we find

$$x_4 = (x^2 - 1)\,x_2 - xx_1 \quad \text{and} \quad x_4 = (x^2 - 1)\,x_1 - xx_2 \qquad (***)$$

Now we equate the right-hand members of equalities (***); this yields

$$(x^2 - 1)\,x_2 - xx_1 = (x^2 - 1)\,x_1 - xx_2$$

whence

$$(x^2 + x - 1)\,x_1 = (x^2 + x - 1)\,x_2$$

In the further course of the solution it is natural to distinguish between the two cases when $x^2 + x - 1 \neq 0$ and when $x^2 + x - 1 = 0$. If $x^2 + x - 1 \neq 0$ then obviously $x_1 = x_2$. Since all the unknowns are involved symmetrically in the given equations it can similarly be shown that in this case $x_2 = x_3$, $x_3 = x_4$, $x_4 = x_5$. Thus, here we have $x_1 = x_2 = x_3 = x_4 = x_5$. The substitution of these values into the original equations yields $x = 2$. If $x^2 + x - 1 = 0$ (that is $x^2 - 1 = -x$ and $x = (-1 \pm \sqrt{5})/2$), system (***) reduces to one equation

$$x_4 = -x\,(x_2 + x_1) \qquad (****)$$

We can verify directly that in this case the values $x_3 = xx_2 - x_1$, $x_4 = -x(x_1 + x_2)$ and $x_5 = xx_1 - x_2$ satisfy the given system for arbitrary x_1 and x_2.

Answer: (1) $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ and x is arbitrary; (2) the values $x_1 = x_2 = x_3 = x_4 = x_5$ are arbitrary and $x = 2$; (3) x_1 and x_2 are arbitrary, $x_3 = xx_2 - x_1$, $x_4 = -x(x_1 + x_2)$, $x_5 = xx_1 - x_2$ and $x = (-1 \pm \sqrt{5})/2$.

253. Let x, y, z and t be the sought-for numbers and let $xyzt = A$. It should be noted that $A \neq 0$ because if, for instance, $x = 0$ then the conditions of the problem imply inconsistent equalities $y = z = t = 2$ and $yzt = 2$. Further, the equation $x + yzt = 2$ can be rewritten as

$$x + \frac{A}{x} = 2, \quad \text{that is} \quad x^2 - 2x + A = 0$$

We similarly obtain

$$y^2 - 2y + A = 0, \quad z^2 - 2z + A = 0 \quad \text{and} \quad t^2 - 2t + A = 0$$

For a given A the equation $x^2 - 2x + A = 0$ can have only *two* distinct roots; therefore among the numbers x, y, z and t there are *not more than two different numbers*. Let us consider separately the cases that can take place here.

1°. If $x = y = z = t$ then the given equations yield

$$x + x^3 = 2$$

whence

$$x^3 + x - 2 = 0$$

and, finally,

$$(x - 1)(x^2 + x + 2) = 0$$

From the last equation we find $x_1 = 1$, $x_{2,3} = (-1 \pm \sqrt{-7})/2$. Thus, in this case we have a single real solution: $x = y = z = t = 1$.

2°. If $x = y = z$ while t may be different from these numbers then the conditions of the problem imply

$$x + x^2 t = 2 \quad \text{and} \quad t + x^3 = 2 \qquad (*)$$

On subtracting one of equalities (*) from the other we obtain

$$x^3 - x^2 t - x + t = 0, \quad \text{that is} \quad (x - t)(x^2 - 1) = 0$$

and therefore either $x = t$ (this case has already been investigated) or $x = \pm 1$. For $x = 1$ the first equation (*) immediately yields $t = 1$, that is we again arrive at the solution obtained above. In case $x = -1$ the same equation yields $t = 3$.

3°. If $x = y$ and $z = t$ the system reduces to

$$x + xz^2 = 2, \quad z + x^2 z = 2 \qquad (**)$$

On performing the termwise subtraction of one of these equations from the other we find

$$x - z + xz^2 - x^2 z = 0, \quad \text{that is} \quad (x - z)(1 - xz) = 0$$

Equality $x = z$ immediately leads to case 1°; if $xz = 1$ then (by virtue of the first equation (**)) $x + z = 2$, and we again find the solution $x = z = 1$ obtained earlier.

Answer: Either all the four numbers are equal to 1 or three of them are equal to -1 and the third one is equal to 3.

254. To underline the complete symmetry of the equations forming the given system with respect to the unknowns involved in the system and with respect to the coefficients in these unknowns let us introduce the notation $x = x_1$, $y = x_2$, $z = x_3$, $t = x_4$, $a = a_1$, $b = a_2$, $c = a_3$ and $d = a_4$. Then the given equations take the form

$$\sum_{j=1}^{4} |a_i - a_j| x_j = 1 \quad (i = 1, 2, 3, 4) \qquad (*)$$

Further, let, for instance, $a_1 > a_2 > a_3 > a_4$. Then

$$\left.\begin{aligned} (a_1 - a_2) x_2 + (a_1 - a_3) x_3 + (a_1 - a_4) x_4 &= 1 \\ (a_1 - a_2) x_1 + (a_2 - a_3) x_3 + (a_2 - a_4) x_4 &= 1 \\ (a_1 - a_3) x_1 + (a_2 - a_3) x_2 + (a_3 - a_4) x_4 &= 1 \\ (a_1 - a_4) x_1 + (a_2 - a_4) x_2 + (a_4 - a_3) x_3 &= 1 \end{aligned}\right\} \qquad (**)$$

Let us subtract the second equation of system (**) from the first one, the third equation from the second and the fourth from the third; this results in

$$\left.\begin{aligned}(a_1-a_2)(-x_1+x_2+x_3+x_4)=0\\(a_2-a_3)(-x_1-x_2+x_3+x_4)=0\\(a_3-a_4)(-x_1-x_2-x_3+x_4)=0\end{aligned}\right\}$$

The number a_1, a_2, a_3 and a_4 being pairwise distinct, we must have

$$x_1=x_2+x_3+x_4, \quad x_1+x_2=x_3+x_4, \quad x_1+x_2+x_3=x_4$$

whence it follows that

$$x_2=x_3=0, \quad x_1=x_4$$

and therefore we obtain (for instance, from the last equation (**)) the value

$$x_1=\frac{1}{a_1-a_4}$$

The verification shows that the values $x_2=x_3=0$, $x_1=x_4=$ $=1/(a_1-a_4)$ do in fact satisfy all the equations of the system.

Answer: if $a>b>c>d$ then $x=t=1/(a-d)$, $y=z=0$.

255. Let us denote $x_2-x_1=X_1$, $x_3-x_2=X_2$, $\ldots$, $x_n-x_{n-1}=$ $=X_{n-1}$, $x_1-x_n=X_n$. Then $X_1+X_2+\ldots+X_{n-1}+X_n=0$, and the given system of equations can be rewritten thus:

$$ax_1^2+(b-1)x_1+c=X_1, \quad ax_2^2+(b-1)x_2+c=X_2, \ldots$$
$$\ldots, ax_n^2+(b-1)x_n+c=X_n \qquad (*)$$

It is clear that if the discriminant $\Delta=(b-1)^2-4ac$ of the quadratic binomials on left-hand side of (*) is negative all the binomials retain sign (namely, their signs coincide with that of a). Therefore *all the variables $X_1, X_2, \ldots, X_n$ must be of the same sign as the coefficient a of the equations*, and their sum cannot be equal to zero. This means that the given system *possesses no real solutions*. In case $\Delta=0$ the right-hand sides of equations (*) assume the value 0 only for $x_1=(1-b)/2a$; accordingly, in this case $x_2=x_3=\ldots=x_n=(1-b)/2a$; (for the other values of $x_1, x_2, \ldots, x_n$ the right-hand sides have the same sign as the number a). Therefore the equality $X_1+X_2+\ldots+X_n=0$ is only possible when $X_1=X_2=\ldots=X_n=0$ and $x_1=x_2=\ldots=x_n=$ $=(1-b)/2a$. Finally, for $\Delta>0$ the given system has at least *two* different solutions

$$x_1=x_2=\ldots=x_n=x_1^* \text{ and } x_1=x_2=\ldots=x_n=x_2^*$$

where $x_{1,2}^*=(1-b\pm\sqrt{(1-b)^2-4ac})/2a$.

256. We shall separately investigate the following two cases.

1°. The number n is *even*. On multiplying by one another the 1st, the 3rd, ..., the $(n-1)$th equations of the given system and the 2nd, the 4th, ..., the nth equations we obtain

$$x_1x_2x_3 \dots x_n = a_1a_3a_5 \dots a_{n-1} \quad \text{and} \quad x_1x_2x_3 \dots x_n = a_2a_4a_6 \dots a_n$$

respectively, whence it becomes clear that for $a_1a_3a_5 \dots a_{n-1} \neq a_2a_4a_6 \dots a_n$ the system has *no solutions* at all. If $a_1a_3a_5 \dots a_{n-1} = a_2a_4a_6 \dots a_n$ then, on taking an *arbitrary* value of x_1 (of course, $x_1 \neq 0$), we can consecutively find from the 1st, the 2nd, ..., the $(n-1)$th equations of the system the values

$$x_2 = \frac{a_1}{x_1}, \quad x_3 = \frac{a_2}{x_2}, \quad \dots, \quad x_n = \frac{a_{n-1}}{x_{n-1}}$$

The substitution of all these values into the last equation shows that the last equation is satisfied as well.

2°. The number n is *odd*. On dividing the product of the 1st, the 3rd, ..., the nth equations by the product of the other equations we obtain

$$x_1^2 = \frac{a_1a_3a_5 \dots a_n}{a_2a_4 \dots a_{n-1}} \quad \text{whence} \quad x_1 = \pm\sqrt{\frac{a_1a_3a_5 \dots a_n}{a_2a_4a_6 \dots a_{n-1}}} \qquad (*)$$

(we remind the reader that all the numbers a_i are positive). Further, from the 1st, the 2nd, ..., the $(n-1)$th equations we find in succession

$$x_2 = \frac{a_1}{x_1}, \quad x_3 = \frac{a_2}{x_2}, \quad \dots, \quad x_n = \frac{a_{n-1}}{x_{n-1}}$$

The verification shows that, by virtue of (*), the last equation of the system is also satisfied by these values.

Answer: if n is even and $a_1a_3 \dots a_{n-1} \neq a_2a_4 \dots a_n$ then there are no real solutions; if n is even and $a_1a_3 \dots a_{n-1} = a_2a_4 \dots a_n$, there are infinitely many solutions; if n is odd there are two solutions.

257. (a) First of all we note that if x_0 is a root of the given equation then $-x_0$ is also its root. Consequently, the number of the positive roots coincides with that of the negative roots. Further, the number 0 is a root of the equation, and therefore it suffices to find the number of positive roots. Now we note that if $x/100 = \sin x$ then

$$|x| = 100|\sin x| \leqslant 100 \cdot 1 = 100$$

Hence, the absolute value of a root of the equation cannot exceed 100.

Let us divide the part of the axis Ox from $x = 0$ to $x = 100$ into intervals of length 2π (the last of these intervals may have a smaller length) and determine the number of the roots of the equation lying within each of these intervals (see Fig. 29).

In the first interval (from $x = 0$ to $x = 2\pi$) there is one positive root (and also the root $x = 0$), and each of the following intervals, except the last one, contains two roots. To determine the number of the roots belonging to the last interval let us estimate its length. The number $100/2\pi$ obviously lies between 15 and

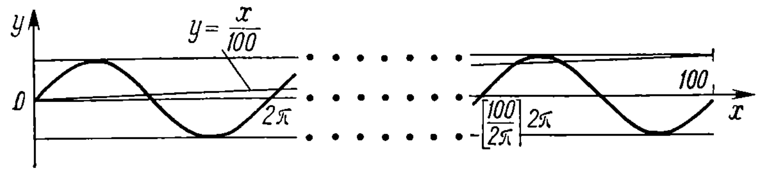


Fig. 29

16 (because $100/15 = 6.66\ldots > 2\pi$ and $100/16 = 6.25 < 2\pi$), and consequently altogether we have 15 intervals of length 2π each and one interval whose length may be less than 2π. The length of this last interval is equal to $100 - 15 \cdot 2\pi > 5 > \pi$, and consequently the horizontal length of the corresponding half wave of the sine curve lying above the x-axis is smaller than the length of that interval, whence it follows that this interval also contains two roots.

Thus, the number of the positive roots of the equation is equal to $1 + 14 \cdot 2 + 2 = 31$. The number of the negative roots is the same, and there is also one root equal to zero.

Finally, the total number of roots is equal to $31 \cdot 2 + 1 = 63$.

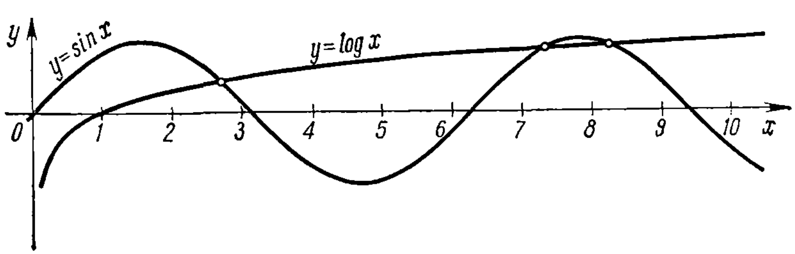


Fig. 30

(b) The solution of this problem is analogous to that of the foregoing problem. It is quite evident that if $\sin x = \log x$ then $x \leqslant 10$ (because, if otherwise, the left-hand member of the equation would be not greater than 1 while the right-hand member would exceed 1). Since $2 \cdot 2\pi > 10$, the interval of the axis Ox from $x = 0$ to $x = 10$ contains only one wave of the sine curve $y = \sin x$ and a part of the next wave (see Fig. 30). The graph of the function $y = \log x$ obviously intersects the first wave of the sine curve at one point. Further, since $2\pi + \pi/2 < 10$, for the point $x = 5\pi/2$ we have $\sin x = 1 > \log x$, and consequently, the graph of $y = \log x$ also intersects the first half of the second positive half wave of the sine curve; further, since at the point $x = 10$ we have $\log x = 1 > \sin x$, the graph of $y = \log x$ must intersect the second half of that half wave as well. We thus see that the total number of the roots of the equation $\sin x = \log x$ is equal to *three*.

258. On adding together the left-hand sides of all the given inequalities we obtan the sum of the numbers $a_1, a_2, \ldots, a_{99}, a_{100}$ each of which is multiplied by a coefficient equal to $1 + (-4) + 3 = 0$. Hence, this sum of 100 nonnegative numbers is equal to zero, which is only possible when *all these numbers are equal to zero*. Thus, the given system of inequalities is in fact a system of *equalities* of the form

$$a_1 - 4a_2 + 3a_3 = 0, \quad a_2 - 4a_3 + 3a_4 = 0, \ \ldots, \ a_{100} - 4a_1 + 3a_2 = 0$$

This system can also be written as

$$a_1 - a_2 = 3(a_2 - a_3), \quad a_2 - a_3 = 3(a_3 - a_4), \quad a_3 - a_4 = 3(a_4 - a_5), \quad a_{99} - a_{100} = 3(a_{100} - a_1),$$
$$a_{100} - a_1 = 3(a_1 - a_2)$$

Now we consecutively find

$$a_1 - a_2 = 3(a_2 - a_3) = 3^2(a_3 - a_4) = 3^3(a_4 - a_5) = 3^{99}(a_{100} - a_1) = 3^{100}(a_1 - a_2)$$

The equality $a_1 - a_2 = 3^{100}(a_1 - a_2)$ implies that $a_1 - a_2 = 0$, and therefore we also have $a_2 - a_3 = \frac{1}{3}(a_1 - a_2) = 0$, $a_3 - a_4 = \frac{1}{3}(a_2 - a_3) = 0, \ \ldots, \ a_{100} - a_1 = \frac{1}{3}(a_{99} - a_{100}) = 0$.

Hence, *all the number* $a_1, a_2, \ldots, a_{99}, a_{100}$ *are equal to one another*; therefore if $a_1 = 1$ then we also have $a_2 = a_3 = \ldots = a_{100} = 1$.

259. *First solution.* Let us rewrite the given inequalities as

$$\left.\begin{aligned} A &= -a - b + c + d > 0 \\ B &= ab - ac - ad - bc - bd + cd > 0 \\ C &= abc + abd - acd - bcd > 0 \end{aligned}\right\} \quad (*)$$

and consider the equation

$$P(x) = (x - a)(x - b)(x + c)(x + d) = x^4 + Ax^3 + Bx^2 + Cx + abcd = 0$$

Since all the coefficients of this equation are positive (because, according to inequalities (*), A, B, and C are positive and the numbers a, b, c and d are also positive, the equation has no positive roots (for $x > 0$ we have $P(x) = x^4 + Ax^3 + Bx^2 + Cx + abcd > 0$). On the other hand, the equation $Px = 0$ has even two positive roots: $x = a$ and $x = b$. Thus, we have arrived at a contradiction, which proves the required proposition.

Second solution. From the first two inequalities indicated in the condition of the problem it follows that

$$(a+b)^2(c+d) < (c+d)(ab+cd)$$

that is

$$(a+b)^2 < ab+cd \qquad (**)$$

(because $c + d > 0$). Similarly, from the last two inequalities we derive

$$(a+b)^2(c+d)\,cd < (ab+cd)(c+d)\,ab$$

that is

$$(a+b)^2\,cd < (ab+cd)\,ab \qquad (***)$$

Further, since $(a+b)^2 - 4ab = (a-b)^2 \geqslant 0$, we have $(a+b)^2 \geqslant 4ab$, and therefore inequalities (**) and (***) imply that

$$4ab < ab + cd \quad \text{and} \quad 4abcd < (ab+cd)\,ab$$

that is

$$cd > 3ab \quad \text{and} \quad 4cd < ab + cd \quad \text{whence} \quad ab > 3cd$$

However, this is impossible because the inequalities $cd > 3ab$ and $cd < 1/3ab$ cannot hold simultaneously.

260. It is evident that

$$\Big(2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n \text{ radical signs}}\Big)\times$$

$$\times\Big(2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n \text{ radical signs}}\Big) = 2^2 -$$

$$-\Big(2+\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1 \text{ radical signs}}\Big) = 2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1 \text{ radical signs}}$$

and therefore the fraction indicated in the condition of the problem is equal to the reciprocal of the expression

$$R_n = 2+\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1 \text{ radical signs}}$$

Hence, it only remains to prove that $R_n < 4$. To this end we shall use the *method of mathematical induction*: it is clear that $R_1 = 2 < 4$ and that

$$\text{if} \quad R_{n-1} < 4 \quad \text{then also} \quad R_n = 2 + \sqrt{R_{n-1}} < 2 + \sqrt{4} = 4$$

Remark. It is easily seen that $\lim\limits_{n\to\infty} R_n = 4$; indeed, the sequence R_n *possesses a limit* because it is bounded ($R_n < 4$ for all n) and increases monotonically (since $R_{n+1} > R_n$ because R_{n+1} is obtained from R_n by replacing 2 in the last radical in the expression of R_n by the greater number $2+\sqrt{2}$). On putting $\lim\limits_{n\to\infty} R_n = R = r^2$ and making n tend to infinity in the relation $R_{n+1} = 2+\sqrt{R_n}$ we obtain in the limit the equality $r^2 = 2+r$, that is $(2-r)(1+r) = 0$ whence $r = 2$ (because $r < 0$), which implies $R = 4$. It follows that for $n\to\infty$ the fraction indicated in the condition of the problem has a limit equal to 1/4, and therefore *the estimate given in the condition of the problem cannot be made more precise.*

261. Let a, b and c be the given numbers. Since $abc = 1$, we have $c = 1/ab$. The second condition of the problem implies

$$a+b+c > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \quad \text{that is} \quad a+b+\frac{1}{ab} > \frac{1}{a}+\frac{1}{b}+ab \qquad (*)$$

Inequality (*) can be brought to the form

$$ab-a-b+1 < \frac{1}{ab}-\frac{1}{a}-\frac{1}{b}+1$$

whence

$$(a-1)(b-1) < \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right) = \frac{1}{ab}(a-1)(b-1)$$

Thus, inequality (*) is equivalent to the inequality

$$\frac{1}{ab}(a-1)(b-1) > (a-1)(b-1),$$

$$\text{that is} \quad (a-1)(b-1)\left(\frac{1}{ab}-1\right) > 0$$

Therefore among the triple $a-1$, $b-1$ and $1/ab-1 = c-1$ there is an even number of negative members; in other words, two of these differences are negative and one is positive, which is what we had to prove. (It is obvious that *all* the three differences cannot be simultaneously positive because if $a > 1$ and $b > 1$ then there must necessarily be $c = 1/ab < 1$, whence it follows that the difference $1/ab - 1$ is negative.)

262. It is clear that the numbers 1959 and 1000 occur accidentally in the condition of the problem; a more general proposition to be proved reads: *if* $a_i > 0$ *for all* $i = 1, 2, \ldots, n$ *and* $\sum a_i = a_1+a_2+\ldots+a_n = 1$ *then the sum* $S_{n,k} = \sum\limits_{i_1, i_2, \ldots, i_k=1}^{n} a_{i_1}a_{i_2}\ldots a_{i_k}$ *of all the possible products of k factors chosen from the n given numbers* $a_1, a_2, \ldots, a_n$ (*where* $1 \leqslant k < n$) *is not greater than* 1 *and for* $k > 1$ *this sum is less than* 1. To prove this proposition we can, for instance, use the *induction method* (with respect to the numbes n and k). It is evident that the proposition is true for $n=1$ and for $n=2$. Let us assume that the proposition has already been proved for all n smaller than a certain value N and that for the value N itself the proposition has been proved for all $k \geqslant 2$ smaller than a fixed value K $(K \geqslant 2)$. (It is obvious that for any n and $k=1$ the proposition is true.) Our aim is to show that under this assumption the proposition is true for the values N and K themselves.

Let us consider the sum $S_{N,K} = \sum_{i_1, i_2, \ldots, i_K=1}^{N} a_{i_1} a_{i_2} \ldots a_{i_K}$. It can be written in the form

$$S_{N,K} = \sum_{i_1, i_2, \ldots, i_{K-1}=1}^{N-1} a_{i_1} a_{i_2} \ldots a_{i_{K-1}} a_N + \sum_{i_1, i_2, \ldots, i_K=1}^{N-1} a_{i_1} a_{i_2} \ldots a_{i_K} = S_{N-1, K-1} \cdot a_N + S_{N-1, K}$$

where $S_{N-1, K-1}$ and $S_{N-1, K}$ are, respectively, the sum of all possible products of $K-1$ factors and the sum of the products of K factors chosen from the numbers $a_1, a_2, \ldots, a_{N-1}$. The sum of these $N-1$ numbers can be written as

$$a_1 + a_2 + \ldots + a_{N-1} = (a_1 + a_2 + \ldots + a_{N-1} + a_N) - a_N = 1 - a_N$$

Next we replace the numbers $a_1, a_2, \ldots, a_{N-1}$ by the numbers $a'_1 = a_1/(1-a_N)$, $a'_2 = a_2/(1-a_N)$, $\ldots$, $a'_{N-1} = a_{N-1}/(1-a_N)$ respectively, the sum of the new numbers being equal to 1. Let us denote the sum of all possible products of $K-1$ factors and the sum of the products of K factors chosen from these new $N-1$ numbers a'_i $(i = 1, 2, \ldots, N-1)$ as $S'_{N-1, K-1}$ and $S'_{N-1, K}$ respectively. By the hypothesis, $S'_{N-1, K-1} \leqslant 1$ and $S'_{N-1, K} < 1$; on the other hand, since the numbers a'_i are proportional to the numbers a_i $(i = 1, \ldots, N-1)$, we obviously have

$$S_{N-1, K-1} = S'_{N-1, K-1} \cdot (1-a_N)^{K-1} \leqslant (1-a_N)^{K-1}$$

and

$$S_{N-1, K} = S'_{N-1, K} \cdot (1-a_N)^{K} < (1-a_N)^{K}$$

Finally we obtain

$$S_{N,K} = S_{N-1, K-1} \cdot a_N + S_{N-1, K} < (1-a_N)^{K-1} \cdot a_N + (1-a_N)^K = (1-a_N)^{K-1} [a_N + (1-a_N)] = (1-a_N)^{K-1} < 1$$

which completes the proof.

263. For $N = 2$ we have only one pair of numbers m and n satisfying the condition of the problem, namely, $m = 1$, $n = 2$; in this case the "sum" s_2 of the fractions under consideration is equal to $\frac{1}{1\cdot 2} = \frac{1}{2}$. For $N = 3$ we have two such pairs: $m = 1$, $n = 3$ and $m = 2$, $n = 3$; in this case the sum of the fractions under consideration is $s_3 = 1/1\cdot 3 + 1/2\cdot 3 = 1/2$. Let us prove that *the sum s_N is equal to* $1/2$ *for any natural* $N > 1$.

Since the assertion we have stated holds both for $N = 2$ and for $N = 3$ we can use the *method of mathematical induction*. Let us suppose that $s_{N-1} = 1/2$ and prove that then we also have $s_N = 1/2$. It is clear that the sums s_{N-1} and s_N are connected in such a way that to obtain the sum s_N from the sum s_{N-1} we must perform the following operations. On the one hand, we must delete from the sum s_{N-1} all the terms having the form $1/mn$ where $m + n = N$, that is the terms of the form $1/i(N - i)$ (here $1 \leqslant i < N/2$ and the numbers i and $N - i$ are relatively prime), and, on the other hand, we must add to the sum s_{N-1} all the possible fractions of the form $1/jN$ where $1 \leqslant j < N$ and the numbers j and N are relatively prime. For every i we have

$$\frac{1}{i(N-i)} = \frac{1}{iN} + \frac{1}{(N-i)N}$$

and the numbers i and $N - i$ are relatively prime if and only if i and N are relatively prime, that is if and only if $N - i$ and N are relatively prime. Therefore the sum s_N is obtained from the sum s_{N-1} by deleting a number of fractions of the form $1/i(N - i)$ and adding instead of every such fraction a sum of fractions of the form $1/iN + 1/(N - i)N$, this sum "compensating" for the deleted fraction. Consequently $s_N = s_{N-1} = 1/2$.

264. We assert that *all* the given 1973 numbers *are the same*. Indeed, let us suppose that this is not true. For definiteness, let $a_1 = a_2 = a_3 = \ldots = a_i \neq a_{i+1}$. For the sake of simplicity we shall index the given numbers in a cyclic order: let us assign the index 1 to the number a_i, the index 2 to the number a_{i+1} and so on up to the number a_{i-1} inclusive to which we assign the index 1973. Thus, in what follows we shall assume that not all the numbers are equal and that $a_1 \neq a_2$; for definiteness, let $a_1 > a_2$ (the case $a_1 < a_2$ is investigated in a similar way).

From the equalities indicated in the condition of the problem it clearly follows that either all the numbers in question are greater than 1 or all the numbers are less than 1 or, finally, all the numbers are equal to 1. In the last case all the numbers a_k (where $k = 1, 2, \ldots, 1973$) are equal to one another, and therefore it remains to consider the other two cases.

1°. *All the numbers a_k are greater than* 1. In this case

$$a_1^{a_2} = a_2^{a_3} \text{ and } a_1 > a_2 \text{ imply } a_2 < a_3$$
$$a_2^{a_3} = a_3^{a_4} \text{ and } a_2 < a_3 \text{ imply } a_3 > a_4$$
$$a_3^{a_4} = a_4^{a_5} \text{ and } a_3 > a_4 \text{ imply } a_4 < a_5$$
. . . .

. . . .

Thus, we have

$$a_1 > a_2 < a_3 > a_4 < a_5 \ldots a_{1972} < a_{1973} > a_1 < a_2$$

Hence, we have arrived at a contradiction (the inequality $a_1 > a_2$ contradicts the inequality $a_1 < a_2$), which proves that in case 1° we must necessarily have $a_1 = a_2 = a_3 = \ldots = a_{1973}$.

2°. *All the numbers a_k are less than* 1. In this case

$$a_1^{a_2} = a_2^{a_3} \text{ and } a_1 > a_2 \text{ imply } a_2 > a_3$$
$$a_2^{a} = a_3^{a_4} \text{ and } a_2 > a_3 \text{ imply } a_3 > a_4$$
. . . .

. . . .

Thus, here we have

$$a_1 > a_2 > a_3 > a_4 > \ldots > a_{1973} > a_1$$

The contradictory inequality $a_1 > a_1$ we have obtained shows that in this case as well all the numbers a_k must necessarily be equal to one another.

265. The proposition of the problem is true for $n = 1$ and for $n = 2$ because

$$x_1^0 + x_2^0 = 1 + 1 = 2, \quad x_1 + x_2 = 6$$

and

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (6)^2 - 2 \cdot 1 = 34$$

Further, we have

$$x_1^n + x_2^n = (x_1 + x_2)(x_1^{n-1} + x_2^{n-1}) - x_1x_2(x_1^{n-2} + x_2^{n-2}) =$$
$$= 6(x_1^{n-1} + x_2^{n-1}) - 1 \cdot (x_1^{n-2} + x_2^{n-2})$$

that is

$$x_1^n + x_2^n = 5(x_1^{n-1} + x_2^{n-1}) + [(x_1^{n-1} + x_2^{n-1}) - (x_1^{n-2} + x_2^{n-2})] \quad (*)$$

First of all, this formula implies that if $x_1^{n-2} + x_2^{n-2}$ and $x_1^{n-1} + x_2^{n-1}$ are whole numbers then $x_1^n + x_2^n$ is also a whole number, whence, by the *principle of mathematical induction*, it follows that the first assertion of the problem is true.

Now, let n be the *first natural number* such that $x_1^n + x_2^n$ is divisible by 5. From formula (*) it follows that in this case the difference $(x_1^{n-1} + x_2^{n-1}) - (x_1^{n-2} + x_2^{n-2})$ must also be divisible by 5. On replacing n by $n - 1$ in formula (*) we obtain

$$x_1^{n-1} + x_2^{n-1} = 5(x_1^{n-2} + x_2^{n-2}) + (x_1^{n-2} + x_2^{n-2}) - (x_1^{n-3} + x_2^{n-3})$$

whence it follows that the expression

$$x_1^{n-3} + x_2^{n-3} = 5(x_1^{n-2} + x_2^{n-2}) - [(x_1^{n-1} + x_2^{n-1}) - (x_1^{n-2} + x_2^{n-2})]$$

must also be divisible by 5, which contradicts the assumption that $x_1^m + x_2^m$ is not divisible by 5 for *all* the numbers m smaller than n. It follows that a positive integer n for which $x_1^n + x_2^n$ is divisible by 5 cannot exist. It is readily seen that the assertion of the problem holds for the negative integers n as well: if $n < 0$ then the sum

$$x_1^n + x_2^n = \frac{1}{x_1^{-n}} + \frac{1}{x_2^{-n}} = \frac{x_1^{-n} + x_2^{-n}}{(x_1x_2)^{-n}} = x_1^{-n} + x_2^{-n}$$

is an integral number not divisible by 5 because $-n > 0$.

266. Let us suppose that the sum $a_1 + a_2 + \ldots + a_{1000}$ contains n positive terms and $1000 - n$ negative terms. Then all the pairwise products of the n positive terms (the number of these products is obviously equal to $n(n-1)/2$) and all the pairwise products of $1000 - n$ negative terms (the number of such products is equal to $(1000 - n)(1000 - n - 1)/2$) are positive, and the pairwise products of positive terms by negative ones (the number of these products is equal to $n(1000 - n)$) are negative. The condition of the problem requires that the relation

$$\frac{n(n-1)}{2} + \frac{(1000-n)(1000-n-1)}{2} = n(1000-n)$$

should be fulfilled. It follows that

$$\frac{n^2 - n + (1000-n)^2 - (1000-n)}{2} = 1000n - n^2$$

whence

$$2n^2 - 2000n + \frac{999\,000}{2} = 0$$

and

$$n = \frac{1000 \pm \sqrt{1\,000\,000 - 999\,000}}{2} = \frac{1000 \pm \sqrt{1000}}{2}$$

which is obviously impossible.

For the second expression we argue in a similar way and arrive at the condition

$$n=\frac{10\,000\pm\sqrt{10\,000}}{2}=\frac{10\,000\pm 100}{2}$$

It follows that the numbers of the positive and negative duplicated products in the second expression may be equal; to this end it is sufficient that the original polynomial should contain $(10\,000+100)/2=5050$ positive terms and $(10\,000-100)/2=4950$ negative terms or 5050 negative terms and 4950 positive terms.

267. First of all we can write

$$(\sqrt{2}-1)^1=\sqrt{2}-\sqrt{1}$$

and

$$(\sqrt{2}-1)^2=3-2\sqrt{2}=\sqrt{9}-\sqrt{8}$$

Now we shall prove that if the expression

$$(\sqrt{2}-1)^{2k-1}=B\sqrt{2}-A=\sqrt{2B^2}-\sqrt{A^2}$$

can be written in the form $\sqrt{N}-\sqrt{N-1}$, that is if $2B^2-A^2=1$, then the number

$$(\sqrt{2}-1)^{2k+1}=B'\sqrt{2}-A'$$

can also be represented in this form, that is $2B'^2-A'^2=1$. Indeed, we have

$$(\sqrt{2}-1)^{2k+1}=(\sqrt{2}-1)^{2k-1}(\sqrt{2}-1)^2=$$
$$=(B\sqrt{2}-A)(3-2\sqrt{2})=(3B+2A)\sqrt{2}-(4B+3A)$$

and consequently

$$B'=3B+2A \quad \text{and} \quad A'=4B+3A$$

whence

$$2B'^2-A'^2=2(3B+2A)^2-(4B+3A)^2=$$
$$=18B^2+24AB+8A^2-16B^2-24AB-9A^2=2B^2-A^2=1$$

which is what we had to prove.

In just the same way it is proved that if a number $(\sqrt{2}-1)^{2k}=C-D\sqrt{2}$ can be represented in the form $\sqrt{N}-\sqrt{N-1}$ then the number $(\sqrt{2}-1)^{2k+2}=C'-D'\sqrt{2}$ can also be written in that form.

By the principle of mathematical induction, it follows that the assertion of the problem is true.

268. If $(A+B\sqrt{3})^2=C+D\sqrt{3}$ then $C=A^2+3B^2$, $D=2AB$ and $(A-B\sqrt{3})^2=A^2+3B^2-2AB\sqrt{3}=C-D\sqrt{3}$. Consequently, if there were $(A+B\sqrt{3})^2 = 99\,999 + 111\,111\sqrt{3}$ then we should also have $(A-B\sqrt{3})^2 = 99\,999 - 111\,111\sqrt{3}$, which is impossible because $99\,999 - 111\,111\sqrt{3}$ is less than zero while the square of every real number is nonnegative.

269. Let us suppose that $\sqrt[3]{2} = p + q\sqrt{r}$. On raising both members of this equality to the third power we obtain

$$2 = p^3 + 3p^2q\sqrt{r} + 3pq^2r + q^3r\sqrt{r}$$

that is

$$2 = p(p^2 + 3q^2r) + q(3p^2 + q^2r)\sqrt{r}$$

Now let us show that if $\sqrt[3]{2} = p + q\sqrt{r}$ then $\sqrt[3]{2}$ is a *rational* number. Indeed, if $q = 0$ then $\sqrt[3]{2} = p$ is a rational number. If $q \neq 0$ and $3p^2 + q^2r \neq 0$ then the last equality implies

$$\sqrt{r} = \frac{2 - p(p^2 + 3q^2r)}{q(3p^2 + q^2r)}$$

whence

$$\sqrt[3]{2} = p + q\,\frac{2 - p(p^2 + 3q^2r)}{q(3p^2 + q^2r)}$$

that is $\sqrt[3]{2}$ is again a rational number. Finally, if $3p^2 + q^2r = 0$ then

$$q^2r = -3p^2, \quad 2 = p[p^2 + 3(-3p^2)] = -8p^3$$

that is $\sqrt[3]{2} = -2p$ is again a rational number.

Hence, it only remains to show that $\sqrt[3]{2}$ is not a rational number. The proof of this proposition is well known. If we suppose that $\sqrt[3]{2}$ is equal to an irreducible fraction m/n then $2 = m^3/n^3$, that is $m^3 = 2n^3$. Thus, the number m^3 and also the number m are even integers, and consequently the number m^3 is divisible by 8. In this case $n^3 = m^3/2$ must also be even, and consequently the number n is even, which contradicts the assumption that the fraction m/n is irreducible. We see that the assumption that $\sqrt[3]{2} = p + q\sqrt{r}$ leads to a contradiction.

270. Let us denote $(n + \sqrt{n^2 - 4})/2 = x$; then

$$\frac{1}{x} = \frac{2}{n + \sqrt{n^2 - 4}} = \frac{2(n - \sqrt{n^2 - 4})}{4} = \frac{n - \sqrt{n^2 - 4}}{2}$$

and we see that x satisfies the equation $x + 1/x = n$. If $x + 1/x$ is equal to a whole number n then so is the number $x^m + 1/x^m$, which can easily be proved by the *method of mathematical induction*. Indeed, if we assume that all the expressions $a_i = x^i + 1/x^i$ are whole numbers for all $i \leqslant N$ then $a_{N+1} = x^{N+1} + 1/x^{N+1}$ is also a whole number, which is a consequence of the following relation:

$$a_N a_1 = \left(x^N + \frac{1}{x^N}\right)\left(x + \frac{1}{x}\right) = \left(x^{N+1} + \frac{1}{x^{N+1}}\right) + \left(x^{N-1} + \frac{1}{x^{N-1}}\right) = $$
$$= a_{N+1} + a_{N-1}$$

whence

$$a_{N+1} = a_N a_1 - a_{N-1}$$

Thus, denoting $x^m = y$ we can write the equality $y + 1/y = k$ where $k = a_m$ is a natural number. This equality is equivalent to a quadratic equation in y whose solution is $y = (k \pm \sqrt{k^2 - 4})/2$. Since the number $x = (n + \sqrt{n^2 - 4})/2$ exceeds 1, we also have $x^m = y > 1$ whence it follows that $y = x^m = A = (k + \sqrt{k^2 - 4})/2$ and $1/x^m = (k - \sqrt{k^2 - 4})/2 < 1$.

271. First of all we note that *a real number α cannot be represented in two different ways as a sum $\alpha = x + y\sqrt{2}$ where x and y are rational numbers.* For, if $\alpha = a + b\sqrt{2} = a_1 + b_1\sqrt{2}$ (where a, b, a_1 and b_1 are rational numbers) then $\sqrt{2} = (a - a_1)/(b_1 - b)$, which is only possible when $a = a_1$ and $b_1 = b$ because the differences $a - a_1$ and $b_1 - b$ are rational numbers and $\sqrt{2}$ is an irrational number. Further, using Newton's binomial formula, we can write the equality indicated in the condition of the problem in the form

$$(X + Y\sqrt{2}) + (Z + T\sqrt{2}) = 5 + 4\sqrt{2}$$

where

$$X = x^{2n} + C(2n, 2)\, x^{2n-2} \cdot 2y^2 + \ldots;$$
$$Y = C(2n, 1)\, x^{2n-1} y + C(2n, 3)\, x^{2n-3} \cdot 2y^3 + \ldots$$

etc. From this equality it follows that

$$X + Z = 5 \quad \text{and} \quad Y + T = 4 \qquad (*)$$

Now, on multiplying the second equality (*) by $\sqrt{2}$ and subtracting the result from the first equality we obtain

$$(X - Y\sqrt{2}) + (Z - T\sqrt{2}) = 5 - 4\sqrt{2}$$

that is

$$(x - y\sqrt{2})^{2n} + (z - t\sqrt{2})^{2n} = 5 - 4\sqrt{2} \qquad (**)$$

Thus, relation (**) must necessarily hold provided that the equality indicated in the condition of the problem is fulfilled. However, it can readily be seen that equality (**) cannot hold because its left-hand member is positive whereas its right-hand member is negative. It follows that the equality indicated in the condition of the problem *cannot hold* either.

272. It is impossible. Indeed, suppose that we poured water k_1 times from the first barrel into the second barrel using the first scoop and that we poured water from the second barrel into the first one k_2 times using the same scoop. This means that, as a result, we poured $(k_1 - k_2)\sqrt{2} = k\sqrt{2}$ litres of water from the first barrel into the second one where the integral number $k = k_1 - k_2$ may be nonpositive. Similarly, if we poured water from the first barrel into the second barrel l_1 times using the second scoop and if we poured water from the second barrel into the first one l_2 times using the same scoop then, as a result, we poured $(l_1 - l_2)(2-\sqrt{2}) = l\cdot(2-\sqrt{2})$ litres of water from the first barrel into the second barrel where l is an integral number. Therefore the condition of the problem reguires that the equality $k\sqrt{2} + l(2-\sqrt{2}) = 1$ should be fulfilled, that is $(l-k)\sqrt{2} = 2l - 1$ whence $\sqrt{2} = (2l-1)/(l-k)$. Since $\sqrt{2}$ is an *irrational* number the last equality can only hold (for integral values of k and l) when $l - k = 0$ (that is $l = k$) and $2l - 1 = 0$, whence $l = 1/2$, which is impossible because l is an integral number.

273. *First solution.* In the problem it is required to find all *rational* solutions (x, y) (where $y \geqslant 0$) of the equation $3x^2 - 5x + 9 = y^2$ in the two unknowns x and y (cf. the problems in Sec. 5 of the present book). It is evident that there exists one solution of the form $x = 0$, $y = 3$. Let us put $x = x_1$ and $y = y_1 + 3$; then we obtain

$$3x_1^2 - y_1^2 - 5x_1 - 6y_1 = 0$$

For *every* solution (x_1, y_1) different from $(0, 0)$ we have $y_1/x_1 = m/n$ where m and n are *relatively prime integers* $(y_1 = x_1 \times (m/n))$. Consequently, we have

$$3x_1^2 - \frac{m^2}{n^2}x_1^2 - 5x_1 - 6\frac{m}{n}x_1 = 0$$

whence $x_1 = (5n^2 + 6mn)/(3n^2 - m^2)$ because $x \neq 0$. The formula $x = (5n^2 + 6mn)/(3n^2 - m^2)$ gives the full answer to the question stated in the problem (the solution $x = 0$ corresponds to $n = 0$).

Second solution. If (x, y) is a point with rational coordinates belonging to the second-order curve $y^2 = 3x^2 - 5x + 9$ (it is a *hyperbola*) then the ratio $k = (y-3)/x$ is a *rational* number (it can also be infinite). On the other hand, if k is a rational number then the straight line $y = kx + 3$ intersects that curve at two rational points: $(0, 3)$ and $((6k+5)/(3-k^2), (3k^2+5k+9)/(3-k^2))$. It follows that all the points with rational coordinates belonging to the curve correspond to the values of x expressed by the formula $x = (6k+5)/(3-k^2)$.

274. Let $x^2 + px + q = 0$ and $y^2 + py + q_1 = 0$ be the original and the "rounded" equations respectively where $|q_1 - q| = |\varepsilon| \approx 0.01$. On subtracting the second equation from the first one we obtain $(x^2 - y^2) + p(x - y) = q_1 - q = \varepsilon$, that is $(x - y)(x + y + p) = \varepsilon$. Let us denote as x_1 and y_1 the roots of the two equations which are close to each other and let us denote as x_2 the second root of the first of these equations; then we have $-(x_1 + x_2) = p$, and from the relation that was established above it follows that

$$|y_1 - x_1| = \frac{|\varepsilon|}{|x_1 + y_1 + p|} \approx \frac{|\varepsilon|}{|2x_1 - (x_1 + x_2)|} = \frac{|\varepsilon|}{|x_1 - x_2|} = \frac{|\varepsilon|}{\Delta}$$

which is what we intended to prove.

275. It is clear that if among the given numbers there are several integers then they can simply be discarded because the difference between a sum of any number of rounded numbers and the sum of the numbers themselves does not change when we add to the original set of numbers some more whole numbers while the sum of the numbers itself increases. Therefore if the assertion of the problem is true for *non-integral* numbers then it is also true for any numbers. Now let us denote the given numbers as $a_1, a_2, \ldots, a_n$, their *integral parts* as $[a_1], [a_2], \ldots, [a_n]$ and their *fractional parts* $\{a_i\} = a_i - [a_i]$ (where $i = 1, 2, \ldots, n$) as $\alpha_1, \alpha_2, \ldots, \alpha_n$ (cf. page 37). Let us agree to arrange the numbers so that their fractional parts do not decrease:

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n < 1$$

Now let us round the first k numbers a_i (here $1 \leqslant i \leqslant k$; the choice of the number k where $0 \leqslant k \leqslant n$ will be specified later) replacing them by the numbers $[a_i]$ which are *smaller* than the numbers a_i $(i = 1, 2, \ldots, n)$. As to the $n - k$ remaining numbers a_j (where $k < j \leqslant n$), we shall replace them by the corresponding *greater* numbers $[a_j] + 1$. It is clear that the error appearing when the sum of the numbers is replaced by the sum of the corresponding rounded numbers is the greatest for the sums $a_1 + a_2 + \ldots + a_k$ and $a_{k+1} + a_{k+2} + \ldots + a_n$; for the first sum it is equal to

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k \leqslant k\alpha_k$$

and for the second sum it is equal to

$$(1 - \alpha_{k+1}) + (1 - \alpha_{k+2}) + \ldots + (1 - \alpha_n) \leqslant (n - k)(1 - \alpha_{k+1})$$

Thus, the condition of the problem will be fulfilled if k is such that

$$k\alpha_k \leqslant \frac{n+1}{4} \quad \text{and} \quad (n - k)(1 - \alpha_{k+1}) \leqslant \frac{n+1}{4}$$

Now let k be the *greatest* of the whole numbers satisfying the condition $k\alpha_k \leqslant (n+1)/4$, that is $\alpha_k \leqslant (n+1)/4k$ (this number k may be equal to 0); the stipulation that k is the *greatest possible* of such numbers means that if the index k is replaced by $k+1$ we obtain $\alpha_{k+1} > (n+1)/4(k+1)$. The last inequality implies that

$$(n-k)(1-\alpha_{k+1}) < (n-k)\left(1-\frac{n+1}{4(k+1)}\right)$$

Let us check that $(n-k)(1-(n+1)/4(k+1)) \leqslant (n+1)/4$. Indeed, this inequality is equivalent to the inequality

$$(n-k)(4k-n+3) \leqslant (n+1)(k+1)$$

and the last inequality is obviously equivalent to the inequality

$$[n-(2k+1)]^2 \geqslant 0$$

which obviously holds for any n and k.

Remark. From the solution of the problem it can easily be seen that for an odd $n = 2l+1$ the estimate given in the statement of the problem cannot be made more precise (to obtain the corresponding example we can put $a_1 = a_2 = \ldots = a_n = 1/2$; here the optimal variant occurs when l numbers are replaced by zeros and the other $l+1$ numbers are replaced by unities or vice versa). In case the number $n = 2l$ is *even* the quantity $(n+1)/4$ in the condition of the problem can be decreased (why?; by what amount?).

276. It is clear that for $a < 0.001$ the rounded number a_0 corresponding to the number a is equal to 0, and consequently the quotient a_0/a and any decimal approximation of this quotient are equal to 0. Therefore in what follows we shall assume that $a \geqslant 0.001$. Then of course we also have $a_0 \geqslant 0.001$ whereas the "approximation error" $\alpha = a - a_0$ is less than 0.001. Therefore it is clear that the fraction

$$d = \frac{a_0}{a} = \frac{a-\alpha}{a} = 1 - \frac{\alpha}{a}$$

we are interested in lies between 0 and 1: $0 < d \leqslant 1$. Further, since $\alpha < 0.001$ and $a_0 > 0.001$, it follows that $\alpha < a_0$, and consequently $\alpha + \alpha = 2\alpha < a_0 + \alpha = a$, whence $\alpha/a < 1/2$; therefore

$$d = 1 - \frac{\alpha}{a} > \frac{1}{2} \quad \text{and} \quad d \leqslant 1 \qquad (*)$$

This estimation of the fraction d cannot be made more precise because the ratio $\delta = \alpha/a$ can assume any value within the limits $0 \leqslant \delta < 1/2$ (and consequently d can assume any value within the limits $1/2 < d \leqslant 1$). Indeed, if we put a_0 equal to 0.001, then $\delta = \alpha/(a_0+\alpha) = \alpha/(0.001+\alpha)$ whence

$$\alpha = 0.001\frac{\delta}{1-\delta} \qquad (**)$$

Such a value of α satisfies the inequality $0 \leqslant \alpha < 0.001$ for any δ (where $0 \leqslant \delta < 1/2$) and the number $a = 0.001 + \alpha$ corresponds to the value $d = 1 - \delta$. It is clear that if δ runs over all the values between 0 and 1/2 then d assumes all the admissible values from 1/2 to 1, that is d may assume any of the following values: 0; 0.5; 0.501; 0.502; ... ; 0.999; 1.

277. Let us consider the 1001 numbers

$$0 \cdot \alpha = 0, \quad \alpha, \quad 2\alpha, \quad 3\alpha, \ldots, \quad 1000\alpha$$

and take the fractional part of each of these numbers (the fractional part of a number is equal to the difference between the given number and the greatest integer not exceeding that number). These fractional parts form a set of 1001 numbers not exceeding 1. Now let us divide the interval of the number axes from 0 to 1 into 1000 equal intervals of length 1/1000 each (we shall agree that the left end of each interval is included into the interval while the right end is not). Our aim is to investigate the distribution of the points representing the above fractional parts over these intervals. Since the number of the intervals is equal to 1000 and the number of the points is equal to 1001, at least one interval contains two points. This means that there exist two unequal numbers p and q (both p and q do not exceed 1000) such that the difference between the fractional parts of the numbers $p\alpha$ and $q\alpha$ is less than 1/1000.

For definiteness, we shall assume that $p > q$. Let us consider the number $(p-q)\alpha = p\alpha - q\alpha$. Since $p\alpha = P + d_1$ and $q\alpha = Q + d_2$ where P and Q are integers and d_1 and d_2 are the fractional parts of $p\alpha$ and $q\alpha$, the number $(p-q)\alpha = (P-Q) + d_1 - d_2$ differs from the integer $P - Q$ by less than 1/1000. This means that the fraction $(P-Q)/(p-q)$ differs from α by less than $0.001 \cdot [1/(p-q)]$.

278. (a) If the number α is less than 1 then $\sqrt{\alpha}$ is also less than 1. Now let us suppose that the decimal representation of the number $\sqrt{\alpha}$ involves less than 100 consecutive nines after the decimal point; this means that $\sqrt{\alpha} < 1 - (1/10)^{100}$. On squaring both members of the last inequality we obtain

$$\alpha < 1 - 2\left(\frac{1}{10}\right)^{100} + \left(\frac{1}{10}\right)^{200}$$

Further, we have

$$1 - 2\left(\frac{1}{10}\right)^{100} + \left(\frac{1}{10}\right)^{200} < 1 - \left(\frac{1}{10}\right)^{100}$$

and therefore $\alpha < 1 - (1/10)^{100}$, which means that the decimal representation of the number α cannot have 100 consecutive nines after the decimal point either.

(b) First of all we note that

$$0.\underbrace{1111\ \dots\ 111}_{100\ \text{ones}}=\frac{1}{9}\cdot 0.\underbrace{9999\ \dots\ 999}_{100\ \text{nines}}=\frac{1}{9}\cdot\left(1-\left(\frac{1}{10}\right)^{100}\right)$$

Hence, we have to estimate the expression $\left(\sqrt{1-(1/10)^{100}}\right)/3$. We shall limit ourselves to the solution of Part (4) of Problem 278 (b) from which follow the results of the other parts of the problem.

As is known, for any $a \leqslant 1$ there holds the inequality $\sqrt{1-a} < 1-a/2$ because $(1-a/2)^2 = 1-a+a^2/4 > 1-a$. Therefore

$$\sqrt{1-\left(\frac{1}{10}\right)^{100}} < 1-\frac{1}{2}\left(\frac{1}{10}\right)^{100} = 0.\underbrace{9999\ \dots\ 999}_{100\ \text{nines}}5$$

To make this estimation more precise we shall find two positive numbers c_1 and c_2 such that

$$1-\frac{1}{2}\left(\frac{1}{10}\right)^{100}-c_1\left(\frac{1}{10}\right)^{200} > \sqrt{1-\left(\frac{1}{10}\right)^{100}} > 1-\frac{1}{2}\left(\frac{1}{10}\right)^{100}-c_2\left(\frac{1}{10}\right)^{200}$$

On squaring all the members of the inequalities

$$1-\frac{1}{2}a-c_1a^2 > \sqrt{1-a} > 1-\frac{1}{2}a-c_2a^2$$

we obtain

$$1+\frac{1}{4}a^2+c_1^2a^4-a-2c_1a^2+c_1a^3 > 1-a > 1+\frac{1}{4}a^2+c_2^2a^4-a-2c_2a^2+c_2a^3$$

Now we subtract the number $1-a$ from all the members and cancel the resulting inequalities by a^2:

$$\left(\frac{1}{4}-2c_1\right)+c_1a+c_1^2a^2 > 0 > \left(\frac{1}{4}-2c_2\right)+c_2a+c_2^2a^2$$

We are interested in the case when $a=(1/10)^{100}$; let us show that for this value of a we can, for instance, put $c_1 = 1/8 + a/100 = 0.125 + (1/10)^{102}$ and $c_2 = 1/8 + a/10 = 0.125 + (1/10)^{101}$. Indeed, for *any* $a > 0$ we have

$$\frac{1}{4}-2\left(\frac{1}{8}+\frac{1}{100}a\right)+\left(\frac{1}{8}+\frac{1}{100}a\right)a+\left(\frac{1}{8}+\frac{1}{100}a\right)^2a^2=\left(\frac{1}{8}-\frac{1}{50}\right)a+\ \dots > 0$$

and, on the other hand, for $a = (1/10)^{100}$ we have

$$\frac{1}{4} - 2\left(\frac{1}{8} + \frac{1}{10}a\right) + \left(\frac{1}{8} + \frac{1}{10}a\right)a + \left(\frac{1}{8} + \frac{1}{10}a\right)^2 a^2 =$$

$$= -\left(\frac{1}{5} - \frac{1}{8}\right)a + \left[\frac{1}{10} + \left(\frac{1}{8} + \frac{1}{10}a\right)^2\right]a^2 < 0$$

because the expression $(1/5 - 1/8)a = (3/40)\cdot(1/10)^{100}$ obviously exceeds the last term of the inequality which is close to $(1/10 + 1/64)a^2 = (37/320)a^2 = (37/320)\cdot(1/10)^{200}$.

Thus, we have

$$1 - \frac{1}{2}\left(\frac{1}{10}\right)^{100} - \left(0.125 + \left(\frac{1}{10}\right)^{102}\right)\left(\frac{1}{10}\right)^{200} > \sqrt{0.\underbrace{999\ldots99}_{100\text{ nines}}} >$$

$$> 1 - \frac{1}{2}\left(\frac{1}{10}\right)^{100} - \left(0.125 + \left(\frac{1}{10}\right)^{101}\right)\left(\frac{1}{10}\right)^{200}$$

The last inequalities can be rewritten as

$$0.\underbrace{9999\ldots999}_{100\text{ nines}}4\underbrace{9999\ldots999}_{99\text{ nines}}87\underbrace{49999\ldots999}_{99\text{ nines}} > \sqrt{1 - \left(\frac{1}{10}\right)^{100}} >$$

$$> 0.\underbrace{9999\ldots999}_{100\text{ nines}}4\underbrace{9999\ldots999}_{99\text{ nines}}87\underbrace{49999\ldots999}_{98\text{ nines}}$$

whence, on dividing by 3, we obtain the relation

$$\sqrt{0.\underbrace{1111\ldots111}_{100\text{ ones}}} \approx 0.\underbrace{3333\ldots333}_{100\text{ threes}}1\underbrace{6666\ldots666}_{100\text{ sixes}}24\underbrace{999\ldots999}_{98\text{ nines}}$$

which is accurate to within 301 decimal places after the decimal point.

279. **(a)** Let us denote 1.00000000004 by α and 1.00000000002 by β. Then the expressions indicated in the condition of the problem take the form $(1+\alpha)/(1+\alpha+\alpha^2)$ and $(1+\beta)/(1+\beta+\beta^2)$. Since $\alpha > \beta$ we obviously have

$$\frac{1+\alpha}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\alpha} < \frac{1}{\beta^2} + \frac{1}{\beta} = \frac{1+\beta}{\beta^2}$$

$$\frac{\alpha^2}{1+\alpha} = 1 : \left(\frac{1+\alpha}{\alpha^2}\right) > 1 : \left(\frac{1+\beta}{\beta^2}\right) = \frac{\beta^2}{1+\beta}$$

$$\frac{1+\alpha+\alpha^2}{1+\alpha} = 1 + \frac{\alpha^2}{1+\alpha} > 1 + \frac{\beta^2}{1+\beta} = \frac{1+\beta+\beta^2}{1+\beta}$$

and, finally,

$$\frac{1+\alpha}{1+\alpha+\alpha^2} = 1 : \left(\frac{1+\alpha+\alpha^2}{1+\alpha}\right) < 1 : \left(\frac{1+\beta+\beta^2}{1+\beta}\right) = \frac{1+\beta}{1+\beta+\beta^2}$$

Thus, the second of the two expressions is greater than the first one.

(b) Let us denote the expressions indicated in the condition of the problem as A and B respectively. We obviously have

$$\frac{1}{A}=1+\frac{a^n}{1+a+a^2+\quad+a^{n-1}}=1+\frac{1}{\frac{1+a+a^2+\ldots+a^{n-1}}{a^n}}=$$

$$=1+\frac{1}{\frac{1}{a^n}+\frac{1}{a^{n-1}}+\quad+\frac{1}{a}}$$

and

$$\frac{1}{B}=1+\frac{1}{\frac{1}{b^n}+\frac{1}{b^{n-1}}+\quad+\frac{1}{b}}$$

It readily follows that $1/A > 1/B$, and consequently $B > A$.

280. We shall proceed from the formula

$$(X-a)^2-(x-a)^2=X^2-x^2-2a(X-x)$$

It implies

$$[(X-a_1)^2+(X-a_2)^2+\quad+(X-a_n)^2]-$$
$$-[(x-a_1)^2+(x-a_2)^2+\quad+(x-a_n)^2]=$$
$$=n(X^2-x^2)-2(a_1+a_2+\quad+a_n)(X-x)$$

If we put $x=(a_1+a_2+\quad+a_n)/n$ in the last expression the resultant number will be nonnegative; indeed, we shall have

$$[(X-a_1)^2+(X-a_2)^2+\quad+(X-a_n)^2]-$$
$$-[(x-a_1)^2+(x-a_2)^2+\quad+(x-a_n)^2]=$$
$$=n(X^2-x^2)-2nx(X-x)=n(X^2-x^2-2Xx+2x^2)=$$
$$=n(X-x)^2\geqslant 0$$

It follows that the sought-for value of x is equal to

$$\frac{a_1+a_2+\ldots+a_n}{n}$$

281. (a) There are only the following three essentially different arrangements:

1°. a_1, a_2, a_3, a_4; in this case

$$\Phi_1=(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_4)^2+(a_4-a_1)^2$$

2°. a_1, a_3, a_2, a_4; in this case

$$\Phi_2=(a_1-a_3)^2+(a_3-a_2)^2+(a_2-a_4)^2+(a_4-a_1)^2$$

3°. a_1, a_2, a_4, a_3; in this case

$$\Phi_3=(a_1-a_2)^2+(a_2-a_4)^2+(a_4-a_3)^2+(a_3-a_1)^2$$

Now it is readily seen that

$$\Phi_3 - \Phi_1 = -2a_2a_4 - 2a_1a_3 + 2a_2a_3 + 2a_1a_4 = 2(a_2 - a_1)(a_3 - a_4) < 0$$

and

$$\Phi_3 - \Phi_2 = -2a_1a_2 - 2a_3a_4 + 2a_2a_3 + 2a_1a_4 = 2(a_3 - a_1)(a_2 - a_4) < 0$$

Consequently, the sought-for arrangement is a_1, a_2, a_4, a_3.

(b) *First solution.* Let us consider the expression

$$\Phi = (a_{i_1} - a_{i_2})^2 + (a_{i_2} - a_{i_3})^2 + \dots + (a_{i_{n-1}} - a_{i_n})^2 + (a_{i_n} - a_{i_1})^2$$

where $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ are the given n numbers arranged in the required order. Let a_{i_α} and a_{i_β} $(\alpha < \beta)$ be some two of these numbers. We assert that *if* a_{i_α} *is greater* (or, conversely, *smaller*) *than* a_{i_β} *then* $a_{i_{\alpha-1}}$ *is greater* (or, respectively, *smaller*) *than* $a_{i_{\beta+1}}$*.

Indeed, if this assertion were not true, that is if we had $(a_{i_\alpha} - a_{i_\beta})(a_{i_{\alpha-1}} - a_{i_{\beta+1}}) < 0$, then the permutation changing the order of the numbers $a_{i_\alpha}, a_{i_{\alpha+1}}, a_{i_{\alpha+2}}, \dots, a_{i_\beta}$ to the opposite would decrease the magnitude of the sum Φ because the difference between the new sum Φ' and the original sum Φ can obviously be written in the form

$$\Phi' - \Phi = -2a_{i_{\alpha-1}}a_{i_\beta} - 2a_{i_\alpha}a_{i_{\beta+1}} + 2a_{i_{\alpha-1}}a_{i_\alpha} + 2a_{i_\beta}a_{i_{\beta+1}} = \\ = 2(a_{i_\alpha} - a_{i_\beta})(a_{i_{\alpha-1}} - a_{i_{\beta+1}})$$

This assertion makes it possible to complete the solution of the problem. Since a cyclic permutation of all the numbers (that is a permutation under which the order of the numbers written *circularly* one after another is retained) does not change the magnitude of the sum Φ, we can assume that a_{i_1} is the smallest of the numbers a_i, that is $i_1 = 1$. From this assumption we can draw the conclusion that a_{i_2} and a_{i_n} are the next two numbers following a_1 in their magnitudes. Indeed, if, for instance, there were $a_{i_\beta} < a_{i_2}$ $(\beta \neq n)$ then we should have $(a_{i_2} - a_{i_\beta})(a_{i_1} - a_{i_{\beta+1}}) < 0$, and if there were $a_{i_\beta} < a_{i_n}$ $(\beta \neq 2)$ then we should have $(a_{i_1} - a_{i_{\beta-1}}) \times \times (a_{i_n} - a_{i_\beta}) < 0$. Since the order of the numbers in the sequence $a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n}, a_{i_1}$ can be changed to the opposite without changing the sum Φ, we can assume that $a_{i_2} < a_{i_n}$, $i_2 = 2$, $i_n = 3$.

* Here we conditionally put $a_{i_0} = a_{i_n}$.

Further, we assert that the numbers a_{i_3} and $a_{i_{n-1}}$ follow in their magnitude the numbers a_{i_1}, a_{i_2} and a_{i_n} which we have already considered. Indeed, if, for instance, there were $a_{i_3} > a_{i_\beta}$ ($\beta \neq 1, 2; n-1, n$) then we should have $(a_{i_3} - a_{i_\beta})(a_{i_2} - a_{i_{\beta+1}}) < 0$. Besides, since there must be $(a_{i_3} - a_{i_{n-1}})(a_{i_2} - a_{i_n}) > 0$, we see that $a_{i_3} < a_{i_{n-1}}$, that is $a_{i_3} = a_4$ and $a_{i_{n-1}} = a_5$.

In just the same way we can show that the numbers a_{i_4} and $a_{i_{n-2}}$ follow in their magnitudes the numbers which we have already considered $a_{i_4} < a_{i_{n-2}}$ (that is $i_4 = 6$, $i_{n-2} = 7$). Similarly, the numbers a_{i_5} and $a_{i_{n-3}}$ follow in the same sense the numbers considered before and $a_{i_5} < a_{i_{n-3}}$ ($i_5 = 8$, $i_{n-3} = 9$) etc. Finally, we arrive at the following arrangement of the numbers:

$$a_1 \begin{array}{l} \nearrow a_2 - a_4 - a_6 - \ldots - a_{n-2} \searrow \\ \searrow a_3 - a_5 - a_7 - \ldots - a_{n-1} \nearrow \end{array} a_n \quad \text{for an even } n = 2k$$

and

$$a_1 \begin{array}{l} \nearrow a_2 - a_4 - a_6 - \ldots - a_{n-1} \\ \qquad\qquad\qquad\qquad\quad | \\ \searrow a_3 - a_5 - a_7 - \ldots - a_n \end{array} \quad \text{for an odd } n = 2k+1$$

(here the lines indicate the order in which the numbers follow one another; for instance, in the case of an even n we have the arrangement $a_1, a_2, a_4, a_6, \ldots, a_{n-2}, a_n, a_{n-1}, \ldots, a_7, a_5, a_3$).

Second solution. If we guess in some way that the sought-for arrangement is of the form indicated at the end of the first solution then the proof can be elaborated using the method of mathematical induction. Indeed, for $n = 4$ the proof is quite simple (see the solution of Problem 281 (a)). Now let us suppose that for an even n we have already proved that the sum Φ_n corresponding to the arrangement of the numbers $a_1 < a_2 < a_3 < \ldots < a_n$ written at the end of the first solution is less than the sum Φ_n' corresponding to any other arrangement. We shall show that this implies that the sum Φ_{n+1} corresponding to the arrangement of the $n+1$ numbers $a_1 < a_2 < a_3 < \ldots < a_n < a_{n+1}$ indicated in the first solution is less than the sum Φ_{n+1}' corresponding to any other arrangement of the $n+1$ numbers. We have

$$\Phi_{n+1} - \Phi_n = (a_n - a_{n+1})^2 + (a_{n+1} - a_{n-1})^2 - (a_n - a_{n-1})^2 =$$
$$= 2a_{n+1}^2 - 2a_n a_{n+1} - 2a_{n-1}a_{n+1} + 2a_{n-1}a_n = 2(a_{n+1} - a_n)(a_{n+1} - a_{n-1})$$

On the other hand, if in the arrangement corresponding to the sum Φ_{n+1}' the number a_{n+1} stands between some numbers a_α and

a_β and if Φ'_n corresponds to the arrangement of the n numbers which is obtained from the arrangement of the $n+1$ numbers leading to the sum Φ'_{n+1} by deleting the number a_{n+1} then

$$\Phi'_{n+1} - \Phi'_n = (a_\alpha - a_{n+1})^2 + (a_{n+1} - a_\beta)^2 - (a_\alpha - a_\beta)^2 =$$
$$= 2a_{n+1}^2 - 2a_\alpha a_{n+1} - 2a_\beta a_{n+1} + 2a_\alpha a_\beta =$$
$$= 2(a_{n+1} - a_\alpha)(a_{n+1} - a_\beta) \geqslant \Phi_{n+1} - \Phi_n$$

Thus, we see that

$$\Phi_{n+1} - \Phi'_{n+1} = [\Phi_n - \Phi'_n] + [(\Phi_{n+1} - \Phi_n) - (\Phi'_{n+1} - \Phi'_n)] \leqslant 0$$

(here the expression in the first brackets is nonpositive according to the induction hypothesis, and the nonpositivity of the expression in the second brackets has already been proved). If the sum

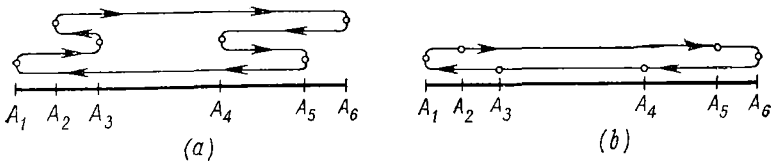


Fig. 31

Φ'_{n+1} differs from Φ_{n+1} then either $\Phi_n - \Phi'_n < 0$ (and, consequently, $\Phi_{n+1} - \Phi'_{n+1} < 0$) or $(\Phi_{n+1} - \Phi_n) - (\Phi'_{n+1} - \Phi'_n) < 0$ (and, consequently, we again have $\Phi_{n+1} < \Phi'_{n+1}$). When n is odd the passage from n to $n+1$ is performed in an analogous manner.

Third solution. We shall also present a simple geometrical solution of the problem. Let us represent the numbers $a_1 < a_2 < a_3 < \ldots < a_n$ as the corresponding points $A_1, A_2, A_3, \ldots, A_n$ on the number line. The line segments A_1A_2, A_2A_3, A_3A_4, $\ldots$, $A_{n-1}A_n$ will be denoted as $d_1, d_2, d_3, \ldots, d_{n-1}$ respectively. Then the sum

$$\Phi = (a_{i_1} - a_{i_2})^2 + (a_{i_2} - a_{i_3})^2 + \quad + (a_{i_{n-1}} - a_{i_n})^2 +$$
$$+ (a_{i_n} - a_{i_1})^2 = A_{i_1}A_{i_2}^2 + A_{i_2}A_{i_3}^2 + \quad + A_{i_{n-1}}A_{i_n}^2 + A_{i_n}A_{i_1}^2$$

is equal to the sum of the squares of the lengths of the segments of the "broken line" $A_{i_1}A_{i_2}A_{i_3} \ldots A_{i_{n-1}}A_{i_n}A_{i_1}$ (whose all segments are in one straight line; see Fig. 31 (a)).

Since this closed broken line covers the whole line segment A_1A_n, each of the line segments $A_kA_{k+1} = d_k$ occurs at least twice in that broken line (it is once passed in the direction from A_k to A_{k+1} and the next time in the opposite direction). Therefore, irrespective of the order of the arrangement of the points, if we express the sum Φ in terms of the line segments $d_1, d_2, \ldots, d_{n-1}$ and open the parentheses, the resultant expression must necessarily involve the term $2d_k^2$, and, consequently, it must involve all the terms $2d_1^2, 2d_2^2, \ldots, 2d_{n-1}^2$. Further, let $A_{k-1}A_k = d_{k-1}$ and $A_kA_{k+1} = d_k$ be two neighbouring line segments. It is evident that if a segment of the broken line covering the line segment A_kA_{k+1} in the direction from A_k to A_{k+1} starts at the point A_k then the segment of the broken line covering A_kA_{k+1} in the opposite direction cannot end at the point A_k. Therefore in all the cases there must exist a segment of the broken line which simultaneously covers the line segments $A_{k-1}A_k$ and A_kA_{k+1}. It follows that in all the cases the sum Φ must involve the term $2d_{k-1}d_k$ and, consequently, all the terms $2d_1d_2, 2d_2d_3, \ldots, 2d_{n-2}d_{n-1}$ as well.

Now it only remains to note that in the case when the arrangement of the points coincides with the one indicated at the end of the first solution we have

$$\Phi = 2d_1^2 + 2d_2^2 + \ldots + 2d_{n-1}^2 + 2d_1d_2 + 2d_2d_3 + \ldots + 2d_{n-2}d_{n-1}$$

(see Fig. 31*b*)). What has been said and the above argument imply that in this case the sum Φ assumes the smallest value.

282. (a) First of all it should be noted that we can assume that all the numbers $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ to be positive; for, if otherwise, we can change the signs of the negative numbers to the opposite; this does not change the left-hand member of the inequality while the right-hand member can only increase.

Let us consider a broken line $A_0A_1A_2 \ldots A_n$ such that the projections of its segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ on the axis Ox are equal to $a_1, a_2, \ldots, a_n$ respectively and the projections of these segments on the axis Oy are equal to $b_1, b_2, \ldots, b_n$; let every vertex of the broken line lie to the right of and higher than the foregoing vertex (Fig. 32). Then Pythagoras' theorem implies

$$A_0A_1 = \sqrt{a_1^2 + b_1^2}, \quad A_1A_2 = \sqrt{a_2^2 + b_2^2}, \ldots, A_{n-1}A_n = \sqrt{a_n^2 + b_n^2},$$
$$A_0A_n = \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + (b_1 + b_2 + \ldots + b_n)^2}$$

whence follows the inequality indicated in the problem.

The length of the broken line $A_0A_1A_2 \ldots A_n$ is *equal* to that of the line segment A_0A_n if and only if all the segments of that broken line are extensions of one another (that is the broken line coincides with a straight line segment). It can easily be seen that this is the case only when $a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n$. In this case only the equality

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \ldots + \sqrt{a_n^2 + b_n^2} =$$
$$= \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + (b_1 + b_2 + \ldots + b_n)^2}$$

holds.

(b) Let us denote by h the altitude of the pyramid and by $a_1, a_2, \ldots, a_n$ the sides of its base ($a_1 + a_2 + \ldots + a_n = P$); accordingly, let $b_1, b_2, \ldots b_n$ be the lengths of the perpendiculars dropped from the foot of the altitude of the pyramid on the sides of the base ($a_1b_1/2 + a_2b_2/2 + \ldots + a_nb_n/2 = S$). Then the lateral area Σ of the pyramid is equal to

$$\frac{1}{2}a_1\sqrt{b_1^2+h^2}+\frac{1}{2}a_2\sqrt{b_2^2+h^2}+\ \ldots\ +\frac{1}{2}a_n\sqrt{b_n^2+h^2}$$

According to the inequality established in Problem 282 (a), we have

$$2\Sigma = \sqrt{(a_1b_1)^2+(a_1h)^2}+\sqrt{(a_2b_2)^2+(a_2h)^2}+\ \ldots$$
$$\ldots + \sqrt{(a_nb_n)^2+(a_nh)^2} \geqslant$$
$$\geqslant \sqrt{(a_1b_1+a_2b_2+\ \ldots\ +a_nb_n)^2+(a_1h+a_2h+\ \ldots\ +a_nh)^2} =$$
$$= \sqrt{4S^2+h^2P^2}$$

where the sign of equality appears only when $a_1b_1 : a_2b_2 : \ldots$ $\ldots : a_nb_n = a_1h : a_2h : \ldots : a_nh$, that is when $b_1 = b_2 = \ldots = b_n$, whence follows the assertion of the problem.

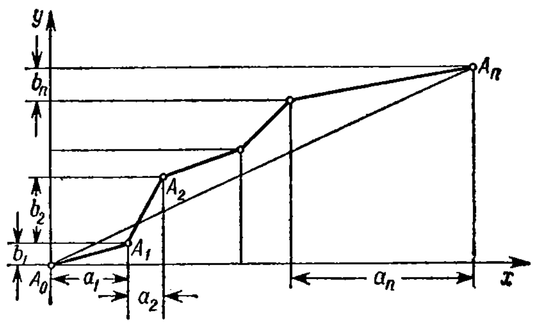


Fig. 32

283. Let us investigate separately the cases when n is even and when n is odd.

1°. *The number n is even.* Let us construct a broken line $A_1A_2A_3 \ldots A_nA_{n+1}A_{n+2}$ such that the lengths of all its segments $A_1A_2, A_2A_3, A_3A_4, \ldots, A_{n+1}A_{n+2}$ are equal to unity and the segments $A_1A_2, A_3A_4, A_5A_6, \ldots, A_{n-1}A_n, A_{n+1}A_{n+2}$ are parallel to one another and are perpendicular to the segments $A_2A_3, A_4A_5, \ldots$ $\ldots, A_nA_{n+1}$ (see Fig. 33; this figure is depicted for the case $n=4$). Further, let us choose a point B_i on each of the line segments A_iA_{i+1} ($i = 1, 2, \ldots, n+1$) or on its extension so that the length of the segment B_iA_{i+1} is equal to a_i (here we put a_{n+1} equal to a_1, that is the point B_{n+1} is chosen in such a way that the equality $B_{n+1}A_{n+2} = a_1$ holds). We shall also agree to take the point B_i to the left of or lower than the point A_{i+1} in case $a_i > 0$ and to the right of or higher than the point A_{i+1} in case $a_i < 0$ (in Fig. 33 we have $0 < a_1 < 1$; $0 < a_2 < 1$; $a_3 > 1$; $a_4 < 0$). Now we draw the broken line $B_1B_2 \ldots B_{n+1}$. By Pythagoras' theorem, we have

$$B_iB_{i+1} = \sqrt{B_iA_{i+1}^2 + B_{i+1}A_{i+1}^2}$$

Since $B_iA_{i+1} = a_i$ and, as can easily be seen, $B_{i+1}A_{i+1} = |1 - a_{i+1}|$, it follows that

$$B_iB_{i+1} = \sqrt{a_i^2 + (1 - a_{i+1})^2}$$

Thus, the sum

$$\sqrt{a_1^2 + (1-a_2)^2} + \sqrt{a_2^2 + (1-a_3)^2} + \ldots + \sqrt{a_{n-1}^2 + (1-a_n)^2} + \sqrt{a_n^2 + (1-a_1)^2}$$

under consideration is equal to the length of the broken line $B_1B_2B_3 \ldots B_{n+1}$.

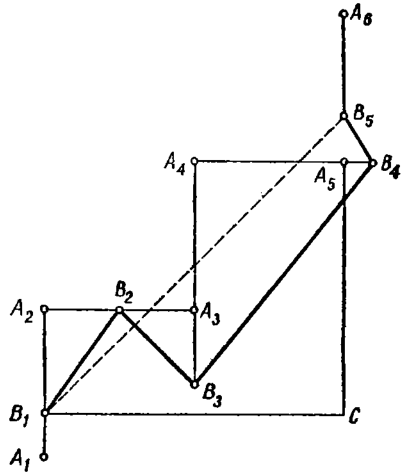


Fig. 33

It is evident that the length of the broken line $B_1B_2B_3 \ldots B_{n+1}$ is always not less than the length of the line segment B_1B_{n+1}. We shall find the length of that segment. To this end let us consider the right triangle B_1CB_{n+1} (see Fig. 33). We see that

$$B_1C = A_2A_3 + A_4A_5 + \ldots + A_nA_{n+1} = \frac{n}{2}$$

and

$$CB_{n+1} = A_1A_2 + A_3A_4 + \ldots + A_{n-1}A_n = \frac{n}{2}$$

(because $A_1B_1 = A_{n+1}B_{n+1} = |1 - a_1|$). These relations imply

$$B_1B_{n+1} = \sqrt{(B_1C)^2 + (CB_{n+1})^2} = \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)^2} = \frac{n\sqrt{2}}{2}$$

whence follows the required inequality.

Now we can easily find in what case the sign $\geqslant$ in this inequality can be replaced by the sign of equality. To this end it is necessary that all the points $B_2, B_3, \ldots, B_n$ should lie on the straight line B_1B_{n+1} (that is the point B_i must coincide with the point of intersection of the straight lines B_1B_{n+1} and A_iA_{i+1}). Since the straight lines B_1B_{n+1} and B_1C form an angle of 45° (because $B_1C = CB_{n+1}$), this condition is fulfilled when

$$B_1A_2 = A_2B_2 = B_3A_4 = A_4B_4 = \ldots = B_{n-1}A_n = A_nB_n$$

that is when $a_1 = (1 - a_2) = a_3 = (1 - a_4) = \ldots = a_{n-1} = (1 - a_n)$. Thus, for an even n the sign of equality appears when

$$a_1 = a_3 = \ldots = a_{n-1} = a \quad \text{and} \quad a_2 = a_4 = \ldots = a_n = 1 - a$$

where a is quite arbitrary.

2°. *The number n is odd* *. Let us put $a_{n+1} = a_1$, $a_{n+2} = a_2, \ldots$ $\ldots, a_{2n} = a_n$ and consider the sum

$$\sqrt{a_1^2 + (1 - a_2)^2} + \sqrt{a_2^2 + (1 - a_3)^2} + \ldots$$
$$\ldots + \sqrt{a_{2n-1}^2 + (1 - a_{2n})^2} + \sqrt{a_{2n}^2 + (1 - a_1)^2}$$

which is obviously equal to twice the sum

$$\sqrt{a_1^2 + (1 - a_2)^2} + \sqrt{a_2^2 + (1 - a_3)^2} + \ldots$$
$$\ldots + \sqrt{a_{n-1}^2 + (1 - a_n)^2} + \sqrt{a_n^2 + (1 - a_1)^2}$$

(each term of the latter sum occurs twice in the former sum). According to what has already been proved, the former sum does not exceed $2n\sqrt{2}/2$, whence it follows that

$$\sqrt{a_1^2 + (1 - a_2)^2} + \sqrt{a_2^2 + (1 - a_3)^2} + \ldots$$
$$\ldots + \sqrt{a_{n-1}^2 + (1 - a_n)^2} + \sqrt{a_n^2 + (1 - a_1)^2} \geqslant \frac{n\sqrt{2}}{2}$$

Thus, we have obtained the required inequality.

In the last inequality the sign of equality appears only when

$$a_1 = a_3 = \ldots = a_{2n-1} = 1 - a_2 = 1 - a_4 = \ldots = 1 - a_{2n}$$

(cf. case 1° above). Now, since $a_1 = a_{n+1}$ and n is odd, the last equality is only possible when

$$a_1 = a_2 = \ldots = a_n = \frac{1}{2}$$

284. *First solution.* Both members of the equality are positive, and therefore, on squaring them, we obtain

$$1 - x_1^2 + 1 - x_2^2 + 2\sqrt{(1 - x_1^2)(1 - x_2^2)} \leqslant 4 - (x_1^2 + 2x_1x_2 + x_2^2)$$

* Let the reader consider as an example the case $n = 3$ to find why the proof presented above for an even n cannot be applied to the case of an odd n.

that is

$$2\sqrt{(1-x_1^2)(1-x_2^2)} \leqslant 2-2x_1x_2$$

whence

$$\sqrt{(1-x_1^2)(1-x_2^2)} \leqslant 1-x_1x_2$$

Now we again square both members to obtain

$$1-x_1^2-x_2^2+x_1^2x_2^2 \leqslant 1-2x_1x_2+x_1^2x_2^2$$

On transposing all the terms to the right we derive the inequality

$$0 \leqslant (x_1-x_2)^2$$

The last inequality is quite evident; the equality sign appears in it only for $x_1 = x_2$. It follows that the original inequality always holds and that it turns into equality only when $x_1 = x_2$.

Second solution. Let us consider a geometrical solution of the problem (analogous solutions can be constructed for any more complicated problems). Let us consider a rectangular Cartesian coordinate system in the plane (see Fig. 34) and construct a circle of unit radius with centre at the origin. The coordinates x and y of the points of the circle are connected by the relation

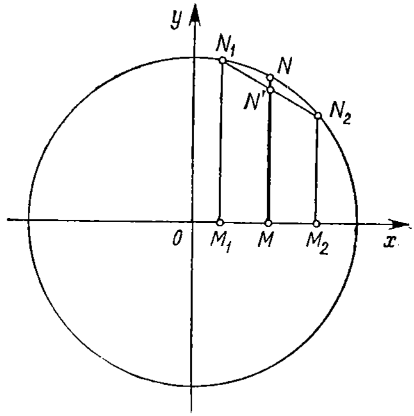


Fig. 34

$$x^2+y^2=1$$

Now let us mark on the x-axis two points M_1 and M_2 with abscissas x_1 and x_2; since $|x_1| \leqslant 1$ and $|x_2| \leqslant 1$, both points lie inside the unit circle or on its boundary. The perpendiculars to the axis of abscissas drawn through these points intersect the upper semi-circle at points N_1 and N_2 (see Fig. 34). We obviously have $M_1N_1 = \sqrt{1-x_1^2}$ and $M_2N_2 = \sqrt{1-x_2^2}$. Since the number $(x_1+x_2)/2$ is equal to the abscissa of the midpoint M of the line segment M_1M_2*, it follows that the quantity $\sqrt{1-[(x_1+x_2)/2]^2}$ is

* This is quite evident when x_1 and x_2 are positive. We can easily verify that the same property remains valid for any signs of x_1 and x_2 as well. (It should be noted however that it is sufficient to prove the inequality indicated in the condition of the problem for positive values of x_1 and x_2 because if x_1

equal to the length of the line segment MN where N is the point of intersection of the circle with the perpendicular to the axis of abscissas drawn through the point M. Further, the sum $M_1N_1 + M_2N_2$ equals twice the length of the midline $N'M$ of the trapezoid $M_1N_1N_2M_2$, that is it is smaller than twice the length of the

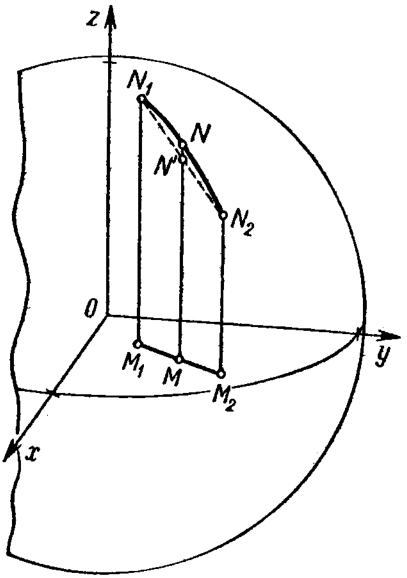


Fig. 35

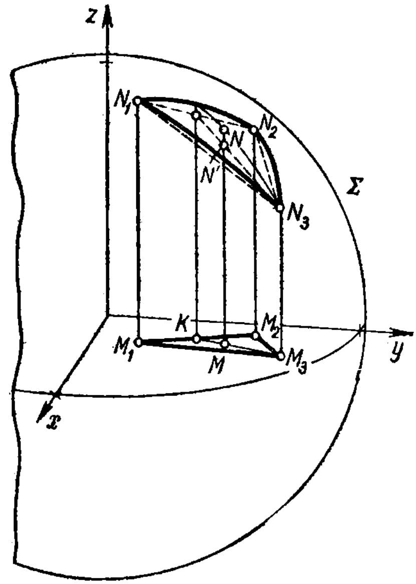


Fig. 36

line segment MN. What has been said proves the required inequality; as is seen from this proof the inequality turns into equality only when the points M_1 and M_2 coincide, that is when $x_1 = x_2$.

Remark. The method used in the the second solution of Problem 284 makes it possible to derive many remarkable inequalities. For instance, let us consider a *sphere* of unit radius with centre at the origin (see Fig. 35). Let M_1 and M_2 be two arbitrary points in the plane Oxy which lie inside the sphere or on its boundary and let N_1 and N_2 be the points of intersection of the sphere with the perpendiculars to the plane Oxy drawn through the points M_1 and M_2. Further, let N and N' be the points of intersection of the perpendicular to the prane Oxy passing through the midpoint M of the line segment M_1M_2 with the sphere and with the line segment N_1N_2 respectively. On denoting the coordinates of the

or x_2 (or both) is nonpositive we can replace the numbers by their absolute values without changing the left-hand member of the inequality while the right-hand member decreases after this replacement).

points M_1 and M_2 as (x_1, y_1) and (x_2, y_2) respectively we can write

$$M_1N_1 = \sqrt{1 - x_1^2 - y_1^2}, \quad M_2N_2 = \sqrt{1 - x_2^2 - y_2^2},$$

$$MN = \sqrt{1 - \left(\frac{x_1 + x_2}{2}\right)^2 - \left(\frac{y_1 + y_2}{2}\right)^2} \text{ and } MN' = \frac{1}{2}(M_1N_1 + M_2N_2)$$

Now, since $MN' \leqslant MN$, it follows that

$$\sqrt{1 - x_1^2 - y_1^2} + \sqrt{1 - x_2^2 - y_2^2} \leqslant 2\sqrt{1 - \left(\frac{x_1 + x_2}{2}\right)^2 - \left(\frac{y_1 + y_2}{2}\right)^2} \quad (*)$$

provided that all the radicands are positive; the sign of equality appears in (*) only in the case when $x_1 = x_2$ and $y_1 = y_2$, that is when the points M_1 and M_2 coincide.

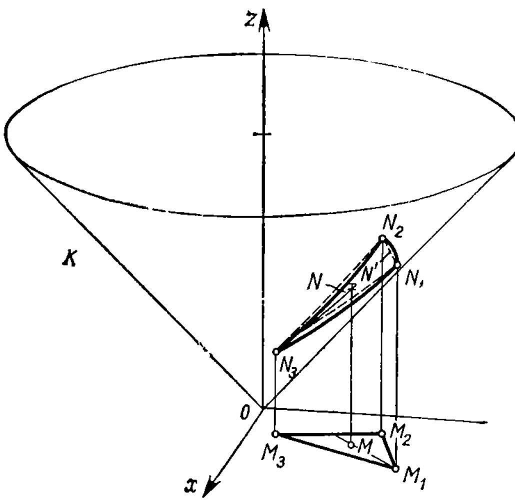


Fig. 37

Similarly, on drawing the perpendiculars to the plane Oxy through the points M_1, M_2 and M_3 and through the point M of intersection of the medians of the triangle $M_1M_2M_3$ (see Fig. 36), we arrive at the inequality

$$\sqrt{1 - x_1^2 - y_1^2} + \sqrt{1 - x_2^2 - y_2^2} + \sqrt{1 - x_3^2 - y_3^2} \leqslant$$
$$\leqslant 3\sqrt{1 - \left(\frac{x_1 + x_2 + x_3}{3}\right)^2 - \left(\frac{y_1 + y_2 + y_3}{3}\right)^2} \quad (**)$$

which means that the length of the line segment MN' does not exceed that of the line segment MN. Inequality (**) also holds in all the cases when all the radicands are positive; the sign of equality appears in (**) only when $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, that is when the points M_1, M_2 and M_3 coincide.

Now let us replace the sphere by the cone whose vertex coincides with the origin and whose axis is Oz, the apex angle of the cone being equal to 90°

(see Fig. 37). Using the same method we obtain the inequality

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\geqslant$$
$$\geqslant 3\sqrt{\left(\frac{x_1+x_2+x_3}{3}\right)^2+\left(\frac{y_1+y_2+y_3}{3}\right)^2} \quad (***)$$

which holds for any x_1, x_2, x_3; y_1, y_2, y_3; the sign of equality appears in (***) only when $x_1/y_1 = x_2/y_2 = x_3/y_3$, that is when the points N_1, N_2 and N_3 lie on one element of the cone.

It should be noted that purely algebraic proofs of inequalities (*), (**) and (***) are extremely complicated.

285. Since $\sin\cos x = -\cos(\pi/2+\cos x)$, there holds the relation

$$\cos\sin x-\sin\cos x=\cos\sin x+\cos\left(\frac{\pi}{2}+\cos x\right)=$$
$$=2\cos\frac{\frac{\pi}{2}+\cos x+\sin x}{2}\cos\frac{\frac{\pi}{2}+\cos x-\sin x}{2}$$

Further,

$$|\cos x+\sin x|=\sqrt{\cos^2 x+2\cos x\sin x+\sin^2 x}=$$
$$=\sqrt{1+\sin 2x}\leqslant\sqrt{2}$$

(we have $|\cos x+\sin x|=\sqrt{2}$ only when $\sin 2x=1$), and similarly

$$|\cos x-\sin x|=\sqrt{\cos^2 x-2\cos x\sin x+\sin^2 x}=$$
$$=\sqrt{1-\sin 2x}\leqslant\sqrt{2}$$

(we have $|\cos x-\sin x|=\sqrt{2}$ only when $\sin 2x=-1$). Since the number $\pi/2\approx 3.14/2=1.57$ is greater than the number $\sqrt{2}\approx 1.41$, it follows that

$$\frac{\pi}{2}>\frac{\frac{\pi}{2}+\cos x+\sin x}{2}>0 \quad\text{and}\quad \frac{\pi}{2}>\frac{\frac{\pi}{2}+\cos x-\sin x}{2}>0$$

Consequently, both expressions $\cos(\pi/2+\cos x+\sin x)/2$ and $\cos(\pi/2+\cos x-\sin x)/2$ are always positive. Thus, the difference $\cos\sin x-\sin\cos x$ is always positive, that is the expression $\cos\sin x$ is greater than $\sin\cos x$ for any x.

286. (a) Let us denote $\log_2\pi=a$ and $\log_5\pi=b$. From the equalities $2^a=\pi$ and $5^b=\pi$ we derive $\pi^{1/a}=2$, $\pi^{1/b}=5$ and $\pi^{1/a}\cdot\pi^{1/b}=2\cdot 5=10$, that is $\pi^{1/a+1/b}=10$. Further, since $\pi^2\approx$ $\approx 3.14^2<10$, we see that the inequality $1/a+1/b>2$ must hold, which is what we had to prove.

(b) Let us denote $\log_2\pi=a$ and $\log_\pi 2=b$. Then $2^a=\pi$ and $\pi^b=2$. The second equality implies $2^{1/b}=\pi$ whence $b=1/a$.

Now we can consecutively rewrite the given inequality in the form

$$\frac{1}{a}+\frac{1}{1/a}>2$$

$$\frac{a^2+1}{a}>2$$

and

$$a^2+1>2a$$

whence $a^2-2a+1=(a-1)^2>0$. The last inequality is quite obvious.

287. *First solution.* Under the condition $\beta>\alpha$ it is required to prove the following inequalities.

(a) $\sin\beta-\sin\alpha<\beta-\alpha$.

We obviously have

$$\sin\beta-\sin\alpha=2\sin\frac{\beta-\alpha}{2}\cos\frac{\beta+\alpha}{2}<2\,\frac{\beta-\alpha}{2}\cdot 1=\beta-\alpha$$

(because for every nonzero angle x lying in the first quadrant there hold the inequalities $\sin x<$ $<x$ and $\cos x<1$ *).

(b) $\tan\beta-\tan\alpha>\beta-\alpha$

We obviously have

$$\beta-\alpha<\tan(\beta-\alpha)=$$
$$=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}<\tan\beta-\tan\alpha$$

(because for every nonzero angle x lying in the first quadrant the inequality $\tan x>x$ holds *).

Second solution. We shall limit ourselves to Problem 286 (a) because Problem 286 (b) can be solved quite analogously.

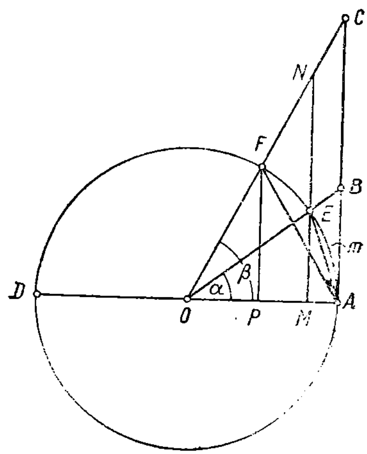


Fig. 38

Let us consider unit circle with centre at a point O (see Fig. 38) and mark the arcs AE and AF equal to α and β respectively. On dropping from the points E and F the perpendiculars EM and FP on the radius OA we can write

$$S_{\triangle OEA}=\frac{1}{2}\sin\alpha$$

$$S_{\triangle OFA}=\frac{1}{2}\sin\beta$$

* E. g., see page 392.

and

$$S_{\text{sector } OEA} = \frac{1}{2}\alpha, \quad S_{\text{sector } OFA} = \frac{1}{2}\beta$$

where the letter S designates the area of a figure.

It follows that

$$\alpha - \sin\alpha = 2\,S_{\text{segment } AmE}$$

and

$$\beta - \sin\beta = 2\,S_{\text{segment } AEF}$$

Consequently,

$$\alpha - \sin\alpha < \beta - \sin\beta$$

288. Let AE and AF be arcs of unit circle with centre at O which are equal to α and β respectively (see again Fig. 38). Let B and C be the points of intersection of the perpendicular to the diameter OA drawn through the point A with the straight lines OE and OF and let M and N be the points of intersection of the perpendicular dropped from the point E on the diameter OA with the straight lines OA and OF. Then we can write

$$S_{\triangle\, OAB} = \frac{1}{2}\tan\alpha, \quad S_{\triangle\, OAC} = \frac{1}{2}\tan\beta$$

and

$$S_{\text{sector } OAE} = \frac{1}{2}\alpha, \quad S_{\text{sector } OAF} = \frac{1}{2}\beta$$

Consequently,

$$\frac{\tan\alpha}{\alpha} = \frac{S_{\triangle\, OAB}}{S_{\text{sector } OAE}} \quad \text{and} \quad \frac{\tan\beta}{\beta} = \frac{S_{\triangle\, OAC}}{S_{\text{sector } OAF}}$$

As is readily seen,

$$\frac{S_{\triangle\, OAB}}{S_{\text{sector } OAE}} < \frac{S_{\triangle\, OBC}}{S_{\text{sector } OEF}}$$

Indeed,

$$\frac{S_{\triangle\, OAB}}{S_{\text{sector } OAE}} < \frac{S_{\triangle\, OAB}}{S_{\triangle\, OEM}}, \quad \frac{S_{\triangle\, OBC}}{S_{\text{sector } OEF}} > \frac{S_{\triangle\, OBC}}{S_{\triangle\, OEN}}$$

and

$$\frac{S_{\triangle\, OAB}}{S_{\triangle\, OEM}} = \frac{S_{\triangle\, OBC}}{S_{\triangle\, OEN}}$$

The inequality $S_{\triangle\, OBC}/S_{\text{sector } OEF} > S_{\triangle\, OAB}/S_{\text{sector } OAE}$ implies

$$\frac{S_{\triangle\, OAB} + S_{\triangle\, OBC}}{S_{\text{sector } OAE} + S_{\text{sector } OEF}} > \frac{S_{\triangle\, OAB}}{S_{\text{sector } OAE}}$$

that is

$$\frac{S_{\triangle\, OAC}}{S_{\text{sector } OAF}} > \frac{S_{\triangle\, OAB}}{S_{\text{sector } OAE}}$$

which is what we intended to prove.

289. Let

$$\arcsin \cos \arcsin x = \alpha$$

The angle α lies within the limits $0 \leqslant \alpha \leqslant \pi/2$ since $0 \leqslant \cos \arcsin x \leqslant 1$ (because $-\pi/2 \leqslant \arcsin x \leqslant \pi/2$). Further, we have $\sin \alpha = \cos \arcsin x$, and consequently

$$\arcsin x = \pm \left(\frac{\pi}{2} - \alpha\right) \quad \text{and} \quad x = \sin\left[\pm\left(\frac{\pi}{2} - \alpha\right)\right] = \pm \cos \alpha$$

Similarly, let $\arccos \sin \arccos x = \beta$; then $0 \leqslant \beta \leqslant \pi/2$ (because $0 \leqslant \sin \arccos x \leqslant 1$ since $0 \leqslant \arccos x \leqslant \pi$) and $\cos \beta = \sin \arccos x$; consequently

$$\arccos x = \frac{\pi}{2} \mp \beta \quad \text{and} \quad x = \cos\left(\frac{\pi}{2} \mp \beta\right) = \pm \sin \beta$$

Now, from the relation $\cos \alpha = \sin \beta = \pm x$ we conclude that

$$\alpha + \beta = \arcsin \cos \arcsin x + \arccos \sin \arccos x = \frac{\pi}{2}$$

290. Let us suppose that the sum

$$\cos 32x + a_{31} \cos 31x + a_{30} \cos 30x + a_{29} \cos 29x + \dots \\ \dots + a_2 \cos 2x + a_1 \cos x \quad (*)$$

assumes only positive values for all x. On replacing x by $x + \pi$ in this sum we arrive at the expression

$$\cos 32(x + \pi) + a_{31} \cos 31(x + \pi) + a_{30} \cos 30(x + \pi) + \\ + a_{29} \cos 29(x + \pi) + \dots + a_2 \cos 2(x + \pi) + \\ + a_1 \cos(x + \pi) = \cos 32x - a_{31} \cos 31x + a_{30} \cos 30x - \\ - a_{29} \cos 29x + \dots + a_2 \cos 2x - a_1 \cos x \quad (**)$$

which must also assume only positive values for all x. Therefore the expression

$$\cos 32x + a_{30} \cos 30x + \dots + a_4 \cos 4x + a_2 \cos 2x$$

which is equal to half the sum of expressions (*) and (**) also assumes only positive values for all x.

Now we replace x by $x + \pi/2$ in the last expression, which results in

$$\cos 32\left(x + \frac{\pi}{2}\right) + a_{30} \cos 30\left(x + \frac{\pi}{2}\right) + \\ + a_{28} \cos 28\left(x + \frac{\pi}{2}\right) + \dots + a_4 \cos 4\left(x + \frac{\pi}{2}\right) + \\ + a_2 \cos 2\left(x + \frac{\pi}{2}\right) = \cos 32x - a_{30} \cos 30x + \\ + a_{28} \cos 28x - \dots + a_4 \cos 4x - a_2 \cos 2x$$

Next we consider the expression

$$\cos 32x + a_{28}\cos 28x + a_{24}\cos 24x + \dots + a_8\cos 8x + a_4\cos 4x$$

which is equal to half the sum of the last two expressions; this expression must also assume only positive values for all x.

On replacing x by $x+\pi/4$ in the last expression and forming half the sum of the resultant expression and the original expression we obtain the sum

$$\cos 32x + a_{24}\cos 24x + a_{16}\cos 16x + a_8\cos 8x$$

Now we replace x by $x+\pi/8$ in this sum and add the resultant expression to the original one; this yields the sum

$$\cos 32x + a_{16}\cos 16x$$

Finally, in just the same way we conclude that the expression

$$\cos 32x$$

must also assume only positive values for all x. However, for $x=\pi/32$ the last expression takes on the value -1. We have thus arrived at a contradiction, which proves the assertion stated in the condition of the problem.

291. We shall proceed from the half-angle formula

$$2\sin\frac{\alpha}{2} = \pm\sqrt{2-2\cos\alpha}$$

where the sign $+$ or $-$ is chosen in accordance with the well-known rule for the sign of the sine function. Using this formula we consecutively determine the sine of the angles

$$a_1 45^\circ;\ \left(a_1+\frac{a_1a_2}{2}\right)\cdot 45^\circ;\quad \left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}\right)\cdot 45^\circ;\ \dots$$
$$\dots;\ \left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}+\dots+\frac{a_1a_2a_3\dots a_n}{2^{n-1}}\right)\cdot 45^\circ$$

Suppose that we have already determined the sine of the angle

$$\left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}+\dots+\frac{a_1a_2\dots a_k}{2^{k-1}}\right)\cdot 45^\circ$$

where $a_1, a_2, a_3, \dots, a_k$ assume values equal to 1 or -1. Since

$$2\left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}+\dots+\frac{a_1a_2\dots a_k}{2^{k-1}}+\frac{a_1a_2\dots a_ka_{k+1}}{2^k}\right)\cdot 45^\circ =$$
$$=\left[\pm 90^\circ \pm\left(a_2+\frac{a_2a_3}{2}+\dots+\frac{a_2a_3\dots a_ka_{k+1}}{2^{k-1}}\right)\cdot 45^\circ\right]$$

where the sign "+" corresponds to $a_1 = +1$ and the sign "—" corresponds to $a_1 = -1$ and since

$$\cos\left[\pm 90^\circ \pm \left(a_2 + \frac{a_2a_3}{2} + \dots + \frac{a_2a_3 \dots a_{k+1}}{2^{k-1}}\right) \cdot 45^\circ\right] =$$
$$= -\sin\left(a_2 + \frac{a_2a_3}{2} + \dots + \frac{a_2a_3 \dots a_{k+1}}{2^{k-1}}\right) \cdot 45^\circ$$

we can now determine the sine of the next angle:

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \dots + \frac{a_1a_2 \dots a_k}{2^{k-1}} + \frac{a_1a_2 \dots a_ka_{k+1}}{2^k}\right) \cdot 45^\circ =$$
$$= \pm\sqrt{2 + 2\sin\left(a_2 + \frac{a_2a_3}{2} + \dots + \frac{a_2a_3 \dots a_{k+1}}{2^{k-1}}\right) \cdot 45^\circ}$$

Now we note that since all the angles under consideration are less than 90° in their absolute values (because even $(1 + 1/2 + +1/4 + \dots +1/2^n) \cdot 45^\circ = 90^\circ - (1/2^n)90^\circ$ is less than 90°) and since the sign of these angles is determined by the sign of a_1, the square root in the last formulas should be taken with the sign plus or minus depending on the sign of a_1. In other words, we can write

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \dots + \frac{a_1a_2 \dots a_k}{2^{k-1}} + \frac{a_1a_2 \dots a_ka_{k+1}}{2^k}\right) \cdot 45^\circ =$$
$$= a_1\sqrt{2 + 2\sin\left(a_2 + \frac{a_2a_3}{2} + \dots + \frac{a_2a_3 \dots a_{k+1}}{2^{k-1}}\right) \cdot 45^\circ}$$

Now let us use the obvious formula

$$2\sin a_1 45^\circ = a_1\sqrt{2}$$

which makes it possible to derive consecutively the following relations:

$$2\sin\left(a_1 + \frac{a_1a_2}{2}\right) \cdot 45^\circ = a_1\sqrt{2 + a_2\sqrt{2}}$$

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{4}\right) \cdot 45^\circ = a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2}}}$$

$$2\sin\left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}+\frac{a_1a_2a_3a_4}{8}\right)\cdot 45^\circ =$$

$$=a_1\sqrt{2+a_2\sqrt{2+a_3\sqrt{2+a_4\sqrt{2}}}}$$

. . . .

$$2\sin\left(a_1+\frac{a_1a_2}{2}+\frac{a_1a_2a_3}{4}+\ldots+\frac{a_1a_2a_3\ldots a_n}{2^{n-1}}\right)\cdot 45^\circ =$$

$$=a_1\sqrt{2+a_2\sqrt{2+a_3\sqrt{2+\ldots+a_n\sqrt{2}}}}$$

which is what we had to prove.

292. Let us suppose that the expansion of the given expression in powers of x is of the form

$$(1-3x+3x^2)^{743}(1+3x-3x^2)^{744}=A_0+A_1x+A_2x^2+\ldots+A_nx^n$$

where $A_0, A_1, A_2, \ldots, A_n$ are the unknown coefficients whose sum we must compute and n is the degree of the polynomial on the right-hand side (it is evident that $n=743\cdot 2+744\cdot 2=2974$). Next we put $x=1$ in this equality, which results in

$$1^{743}\cdot 1^{744}=A_0+A_1+A_2+\ldots+A_n$$

Thus, the sought-for sum is equal to unity.

293. On opening parentheses and collecting terms in the two given expressions we obtain two polynomials in x. Now let us replace x by $-x$ in the given expressions. Then to the two new expressions there correspond two new polynomials which are obtained from the former polynomials by replacing x by $-x$. This means that in each of the new polynomials the coefficients in even powers of x remain the same as in the former polynomials while the signs of the coefficients in the odd powers of x are changed to the opposite. In particular, under this operation the coefficients in x^{20} do not change. Thus, we see that the coefficients in x^{20} in the two former polynomials coincide with the coefficients in x^{20} in the new polynomials obtained after parentheses are opened and like terms are collected in the expressions $(1+x^2+x^3)^{1000}$ and $(1-x^2-x^3)^{1000}$.

It is clear that the first of the new polynomials has a greater coefficient in x^{20} than the second. Indeed, when parentheses are opened in the expression $(1+x^2+x^3)^{1000}$ we obtain only positive coefficients in different powers of x, and when like terms are collected the corresponding coefficients add together. As to the expression $(1-x^2-x^3)^{1000}$, after parentheses are opened in it we obtain coefficients in different powers of x whose absolute values are the same as the absolute values of the coefficients in the first of the new polynomials but the signs of the coefficients may be different; therefore the resultant coefficients appearing after like terms are collected are less than before.

Thus, after parentheses are opened and terms are collected in the expressions $(1+x^2-x^3)^{1000}$ and $(1-x^2+x^3)^{1000}$ the coefficient in x^{20} in the first of the resultant polynomials is greater than that in the other polynomial.

294. The assertion of the problem is a direct consequence of the following transformations:

$$(1-x+x^2-x^3+\dots-x^{99}+x^{100})(1+x+x^2+x^3+\dots$$
$$\dots+x^{99}+x^{100})=[(1+x^2+x^4+\dots+x^{100})-$$
$$-x(1+x^2+x^4+\dots+x^{98})][(1+x^2+x^4+\dots+x^{100})+$$
$$+x(1+x^2+x^4+\dots+x^{98})]=$$
$$=(1+x^2+x^4+\dots+x^{100})^2-x^2(1+x^2+x^4+\dots+x^{98})^2$$

295. **(a)** Using the formula for the sum of a geometric progression and Newton's binomial formula we find

$$(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\dots+x^{1000}=$$
$$=\frac{\frac{x^{1001}}{1+x}-(1+x)^{1000}}{\frac{x}{1+x}-1}=\frac{x^{1001}-(1+x)^{1001}}{x-1-x}=(1+x)^{1001}-x^{1001}=$$
$$=1+1001x+C(1001,\,2)x^2+C(1001,\,3)x^3+\dots+1001x^{1000}$$

Thus, the sought-for coefficient is equal to $C(1001,50)=\frac{1001!}{50!\cdot 951!}$.

(b) Let us denote the given expression as $P(x)$. Then we can write

$$1+x)P(x)-P(x)=$$
$$=[(1+x)^2+2(1+x)^3+\dots+999(1+x)^{1000}+1000(1+x)^{1001}]-$$
$$-[(1+x)+2(1+x)^2+3(1+x)^3+\dots+1000(1+x)^{1000}]=$$
$$=1000(1+x)^{1001}-[(1+x)+(1+x)^2+(1+x)^3+\dots+(1+x)^{1000}]=$$
$$=1000(1+x)^{1001}-\frac{(1+x)^{1001}-(1+x)}{1+x-1}=$$
$$=1000(1+x)^{1001}-\frac{(1+x)^{1001}-(1+x)}{x}$$

It follows that

$$P(x)=\frac{1000(1+x)^{1001}}{x}-\frac{(1+x)^{1001}-(1+x)}{x^2}=$$
$$=1000[1001+C(1001,2)x+C(1001,3)x^2+\dots+1001x^{999}+x^{1000}]-$$
$$-[C(1001,2)+C(1001,3)x+C(1001,4)x^2+\dots+1001x^{998}+x^{999}]$$

Thus, the sought-for coefficient is equal to

$$1000C(1001, 51) - C(1001, 52) = \frac{1000 \cdot 1001!}{51! \cdot 950!} - \frac{1001!}{52! \cdot 949!} =$$

$$= \frac{1001!}{52! \cdot 950!}[52 \cdot 1000 - 950] = \frac{51\,050 \cdot 1001!}{52! \cdot 950!}$$

296. Let us first of all determine the constant term which is obtained after parentheses are opened and like terms are collected in the expression

$$\underbrace{(\ldots((x-2)^2-2)^2 - \ldots - 2)^2}_{k \text{ times}}$$

This term is equal to the value which the expression assumes for $x = 0$, that is it is equal to

$$\underbrace{(\ldots(((-2)^2-2)^2-2)^2 - \ldots - 2)^2}_{k \text{ times}} =$$

$$= \underbrace{(\ldots((4-2)^2-2)^2 - \ldots - 2)^2}_{k-1 \text{ times}} =$$

$$= \underbrace{(\ldots((4-2)^2-2)^2 - \ldots - 2)^2}_{k-2 \text{ times}} = \ldots$$

$$\ldots = ((4-2)^2-2)^2 = (4-2)^2 = 4$$

Now let us denote by A_k the coefficient in x, by B_k the coefficient in x^2 and by P_kx^3 the sum of the terms involving x to the powers higher than 2. Then we can write

$$\underbrace{(\ldots((x-2)^2-2)^2 - \ldots - 2)^2}_{k \text{ times}} = P_kx^3 + B_kx^2 + A_kx + 4$$

On the other hand

$$\underbrace{(\ldots(((x-2)^2-2)^2-2)^2 - \ldots - 2)^2}_{k \text{ times}} =$$

$$= \underbrace{((\ldots((x-2)^2-2)^2 - \ldots - 2)^2}_{k-1 \text{ times}} - 2)^2 =$$

$$= [(P_{k-1}x^3 + B_{k-1}x^2 + A_{k-1}x + 4) - 2]^2 =$$

$$= (P_{k-1}x^3 + B_{k-1}x^2 + A_{k-1}x + 2)^2 =$$

$$= P_{k-1}^2x^6 + 2P_{k-1}B_{k-1}x^5 + \left(2P_{k-1}A_{k-1} + B_{k-1}^2\right)x^4 +$$

$$+ (4P_{k-1} + 2B_{k-1}A_{k-1})x^3 + \left(4B_{k-1} + A_{k-1}^2\right)x^2 +$$

$$+ 4A_{k-1}x + 4 = \left[P_{k-1}^2x^3 + 2P_{k-1}B_{k-1}x^2 +\right.$$

$$\left. + \left(2P_{k-1}A_{k-1} + B_{k-1}^2\right)x + (4P_{k-1} + 2B_{k-1}A_{k-1})\right]x^3 +$$

$$+ \left(4B_{k-1} + A_{k-1}^2\right)x^2 + 4A_{k-1}x + 4$$

whence

$$A_k = 4A_{k-1}, \quad B_k = A_{k-1}^2 + 4B_{k-1}$$

Since $(x-2)^2 = x^2 - 4x + 4$, we have $A_1 = -4$. Consequently, $A_2 = -4\cdot 4 = -4^2$, $A_3 = -4^3$, ... and, generally, $A_k = -4^k$. Now let us compute B_k:

$$B_k = A_{k-1}^2 + 4B_{k-1} = A_{k-1}^2 + 4\left(A_{k-2}^2 + 4B_{k-2}\right) =$$
$$= A_{k-1}^2 + 4A_{k-2}^2 + 4^2\left(A_{k-3}^2 + 4B_{k-3}\right) =$$
$$= A_{k-1}^2 + 4A_{k-2}^2 + 4^2A_{k-3}^2 + 4^3\left(A_{k-4}^2 + 4B_{k-4}\right) = \ldots$$
$$\ldots = A_{k-1}^2 + 4A_{k-2}^2 + 4^2A_{k-3}^2 + \ldots + 4^{k-3}A_2^2 + 4^{k-2}A_1^2 + 4^{k-1}B_1$$

The substitution of

$$B_1 = 1, \quad A_1 = -4, \quad A_2 = -4^2,$$
$$A_3 = -4^3, \ldots, A_{k-1} = -4^{k-1}$$

into this expression yields

$$B_k = 4^{2k-2} + 4\cdot 4^{2k-4} + 4^2\cdot 4^{2k-6} + \ldots + 4^{k-2}\cdot 4^2 + 4^{k-1}\cdot 1 =$$
$$= 4^{2k-2} + 4^{2k-3} + 4^{2k-4} + \ldots + 4^{k+1} + 4^k + 4^{k-1} =$$
$$= 4^{k-1}\left(1 + 4 + 4^2 + 4^3 + \ldots + 4^{k-2} + 4^{k-1}\right) =$$
$$= 4^{k-1}\frac{4^k - 1}{4-1} = \frac{4^{2k-1} - 4^{k-1}}{3}$$

297. **(a)** *First solution.* The binomial $x^k - 1$ is divisible by $x-1$ for any positive integer k; therefore the division of

$$x + x^3 + x^9 + x^{27} + x^{81} + x^{243} = (x-1) + (x^3-1) +$$
$$+ (x^9-1) + (x^{27}-1) + (x^{81}-1) + (x^{243}-1) + 6$$

by $x-1$ leaves a remainder of 6.

Second solution. Let $q(x)$ and r denote the quotient and the remainder resulting from the division of $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x-1$. Then

$$x + x^3 + x^9 + x^{27} + x^{81} + x^{243} = q(x)(x-1) + r$$

The substitution of $x = 1$ into this equality yields $r = 6$.

(b) By analogy with the second solution of the foregoing problem, let us denote by $q(x)$ the quotient resulting from the division of the given polynomial by $x^2 - 1$, and let $r_1x + r_2$ be the sought-for remainder (the division of a polynomial by a quadratic trinomial leaves a remainder which is a binomial of the first degree). Thus,

$$x + x^3 + x^9 + x^{27} + x^{81} + x^{243} = q(x)(x^2-1) + r_1x + r_2$$

On putting $x = 1$ and $x = -1$ in the last equality we obtain

$$6 = r_1 + r_2 - 6 = -r_1 + r_2 \quad \text{whence} \quad r_1 = 6, \quad r_2 = 0$$

Thus, the sought-for remainder is equal to $6x$.

298. Let $p(x)$ be the unknown polynomial and let $q(x)$ and $r(x) = ax + b$ be the quotient and the sought-for remainder resulting from the division of that polynomial by $(x-1)(x-2)$:

$$p(x) = (x-1)(x-2)\,q(x) + ax + b \qquad (*)$$

By the condition of the problem, we have

$$p(x) = (x-1)\,q_1(x) + 2 \quad \text{whence} \quad p(1) = 2$$

and

$$p(x) = (x-2)\,q_2(x) + 1 \quad \text{whence} \quad p(2) = 1$$

Now we substitute $x = 1$ and $x = 2$ into equality (*), which results in

$$2 = p(1) = a + b$$

and

$$1 = p(2) = 2a + b$$

whence

$$a = -1 \quad \text{and} \quad b = 3$$

Thus, the sought-for remainder is equal to $-x + 3$.

299. The polynomial $x^4 + x^3 + 2x^2 + x + 1$ can be factored as $(x^2+1)(x^2+x+1)$. It readily follows that this polynomial is a divisor of the polynomial

$$x^{12} - 1 = (x^6-1)(x^6+1) = (x^3-1)(x^3+1)(x^2+1)(x^4-x^2+1)$$

Namely,

$$x^4 + x^3 + 2x^2 + x + 1 = \frac{x^{12}-1}{(x-1)(x^3+1)(x^4-x^2+1)} = \frac{x^{12}-1}{x^8 - x^7 - x^6 + 2x^5 - 2x^3 + x^2 + x - 1}$$

The division of $x^{1951} - 1$ by $x^4 + x^3 + 2x^2 + x + 1$ is equivalent to the division of $x^{1951} - 1$ by $x^{12} - 1$ and the multiplication of the result by $x^8 - x^7 - x^6 + 2x^5 - 2x^3 + x^2 + x - 1$. Further, it is evident that

$$\frac{x^{1951}-1}{x^{12}-1} = x^{1939} + x^{1927} + x^{1915} + x^{1903} + \ldots + x^{19} + x^7 + \frac{x^7-1}{x^{12}-1}$$

(this can easily be shown with the aid of long division of the polynomials arranged in descending powers of x or with the aid of the identity $x^{1951} - 1 = x^7[(x^{12})^{162} - 1] + x^7 - 1$ and the well-known formula for the division of the difference of two even powers by the difference of the bases). It follows that the sought-for coefficient coincides with the coefficient in x^{14} in the product

$$\left(x^{1939}+x^{1927}+\ \dots\ +x^{31}+x^{19}+x^7+\frac{x^7-1}{x^{12}-1}\right)\times$$
$$\times(x^8-x^7-x^6+2x^5-2x^3+x^2+x-1)$$

That coefficient is obviously equal to —1.

300. From the identity indicated in the condition of the problem it follows that the sought-for polynomial $P(x)=P_n(x)$ where n designates the degree of the polynomial) is divisible by x, that is $P_n(x)=xP_{n-1}(x)$ where $P_{n-1}(x)$ is a polynomial of the $(n-1)$th degree. Therefore

$$P(x-1)=(x-1)P_{n-1}(x-1)$$

and, consequently,

$$x(x-1)P_{n-1}(x-1)=xP(x-1)=(x-26)P(x)$$

It follows that $P(x)$ is divisible by $x-1$ as well, that is $P_n(x)=$ $=x(x-1)P_{n-2}(x)$ $(x-1$ is a divisor of the polynomial $P_{n-1}(x)=(x-1)P_{n-2}(x))$. Therefore we have $P(x-1)=$ $=(x-1)(x-2)P_{n-2}(x-1)$ whence

$$(x-1)(x-2)P_{n-2}(x-1)=$$
$$=(x-26)P_n(x)=(x-26)x(x-1)P_{n-2}(x)$$

The last relation implies that $P_n(x)$ is divisible by $x-2$ as well ($x-2$ is a divisor of $P_{n-2}(x)$), and consequently $P_n(x)=$ $=x(x-1)(x-2)P_{n-3}(x)$. On substituting this expression of $P(x)$ into the original relation we similarly conclude that $P(x)$ is divisible by $x-3$ as well, that is $P_n(x)=x(x-1)(x-2)\times$ $\times(x-3)P_{n-4}(x)$, and so on.

Proceeding in this manner we finally arrive at the following expression for the polynomial $P(x)$:

$$P(x)=P_n(x)=x(x-1)(x-2)(x-3)\ \dots\ (x-25)P_{n-26}(x)$$

The substitution of this expression of the polynomial $P(x)$ into the given identity results in

$$x(x-1)(x-2)\ \dots\ (x-26)P_{n-26}(x-1)=$$
$$=(x-26)x(x-1)\ \dots\ (x-25)P_{n-26}(x)$$

whence it follows that the polynomial $P_{n-26}(x)\equiv Q(x)$ of the $(n-26)$th degree satisfies the identity

$$Q(x-1)\equiv Q(x) \qquad (*)$$

It is clear that if $Q(x) = Q_0(x) = c$ (that is if $Q(x)$ is a polynomial of degree zero equal to a constant number) then relation (*) is fulfilled. Let us show that this is the *only* case when it is fulfilled. Indeed, if $Q(x) = Q_k(x) = a_0x^k + a_1x^{k-1} + \dots + a_{k-1}x + a_k$ where $k \geqslant 1$ and $a_0 \neq 0$ then identity (*) has the form

$$a_0(x-1)^k + a_1(x-1)^{k-1} + \dots \equiv a_0x^k + a_1x^{k-1} + \dots$$

On equating the coefficients in x^{k-1} on both sides we obtain, by virtue of Newton's binomial formula, the equality

$$ka_0 + a_1 = a_1, \quad \text{that is} \quad a_0 = 0$$

However, this contradicts the assumption that $a_0 \neq 0$. Thus, we have $k = 0$ and $Q(x) \equiv c$; hence

$$P(x) = cx(x-1)(x-2)\dots(x-25)$$

is a polynomial of the 26th degree.

301. (a) If all the coefficients of the polynomial $P(x)$ are non-negative then all the numbers $s(1)$, $s(2)$, $s(3)$, ... make sense. Let us consider a power of ten (we denote it $N = 10^k$) such that N is greater than all the coefficients a_0, a_1, a_n, ... of the polynomial $P(x)$. Then the number $P(N) = P(10^k)$ obviously starts with the digits with the aid of which the coefficient a_0 of the polynomial is written, then (possibly after a number of zeros) the digits of a_1 follow; then (possibly again after a number of zeros) the digits of a_2 follow etc. up to the digits of the number a_n. Therefore the number $S = s(10^k)$ is equal to the sum of all digits of all numbers $a_0, a_1, \dots, a_n$. As to the quantities $s(10^{k+1})$, $s(10^{k+2})$, ..., they are equal to the same number S, whence it follows that the number S occurs infinitely many times in the sequence $s(1), s(2), \dots$.

(b) It is clear that if the leading coefficient a_0 of the polynomial $P(x)$ is negative then only a finite number of expressions $s(1)$, $s(2)$, $s(3)$, ... make sense (because in this case for all sufficiently large values of x the sign of the polynomial $P(x)$ coincides with that of its leading coefficient a_0; for instance, this follows from the relation $\lim\limits_{x\to\infty} P(x)/a_0x^n = 1$). Thus, it only remains to consider the case $a_0 > 0$. We shall show that *if $a_0 > 0$ then there is a number $M > 0$ such that all coefficients of the polynomial $\bar{P}(x) \equiv P(x+M)$ are positive.* This will imply that the sequence $\bar{s}(1)$, $\bar{s}(2)$, $\bar{s}(3)$, ... of the sums of the digits of the numbers $\bar{P}(1)$, $\bar{P}(2)$, $\bar{P}(3)$, ... contains infinitely many equal numbers and, since we obviously have $\bar{s}(1) = s(M+1)$, $\bar{s}(2) = s(M+2)$, $\bar{s}(3) = s(M+3)$, ..., the sequence $s(1)$, $s(2)$, $s(3)$, ... must also contain infinitely many equal numbers.

Thus, it is sufficient to prove the auxiliary assertion stated above. We have

$$\bar{P}(x) = P(x+M) =$$
$$= a_0(x+M)^n + a_1(x+M)^{n-1} + a_2(x+M)^{n-2} + \ldots$$
$$\ldots + a_{n-1}(x+M) + a_n = \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \bar{a}_2 x^{n-2} + \ldots + \bar{a}_{n-1}x + \bar{a}_n$$

and therefore, by Newton's binomial formula,

$$\bar{a}_i = a_0 \cdot C(n, i) M^{n-i} + a_1 \cdot C(n-1, i) M^{n-i-1} + \ldots + a_i$$

where $i = 0, 1, 2, \ldots, n$. Hence, $\bar{a}_i$ has the form of a polynomial of the $(n-i)$th degree in the variable M with leading coefficient $a_0 \cdot C(n, i) > 0$. Therefore all the numbers a_i (where $i = 1, 2, \ldots$; it should be noted that $\bar{a}_0 = a_0 > 0$) are positive for sufficiently large M, which we had to prove.

302. Let a_0 and b_0 be the constant terms of the polynomials $f(x)$ and $g(y)$ (that is $f(x) \equiv a_0 + a_1x + \ldots + a_nx^n$ and $g(y) = b_0 + b_1y + \ldots + b_my^m$). Let us put the variable x equal to 0 in the identity $x^{200}y^{200} + 1 \equiv f(x)g(y)$. This yields $a_0 g(y) \equiv 1$, that is $g(y) = 1/a_0$; thus, $g(y)$ is equal to $1/a_0$ for *all* y, which means that $g(y)$ is a constant, that is a polynomial of degree zero. The relation $f(x) = 1/b_0$ is proved similarly; therefore $f(x)g(y) = 1/a_0b_0 \neq x^{200}y^{200} + 1$. We have arrived at a contradiction, which proves the assertion of the problem.

303. Since the (quadratic) equation $p(x) = ax^2 + bx + c = x$ has no real roots, the quadratic trinomial $p(x) - x = ax^2 + (b-1)x + c$ assumes values of one sign for all x, say $p(x) - x > 0$ for all x. Then we have $p(p(x_0)) - p(x_0) > 0$ for any $x = x_0$, that is $p(p(x_0)) > p(x_0)$. By the hypothesis, $p(x_0) - x_0 > 0$, that is $p(x_0) > x_0$, and hence $p(p(x_0)) > x_0$; therefore x_0 *cannot* be a root of the 4th-degree equation $p(p(x)) = x$.

304. Let us assume that $a \geqslant 0$ (if otherwise, we can replace the polynomial $p(x)$ by the polynomial $-p(x) = -ax^2 - bx - c$ satisfying the same conditions). We shall also assume that $b \geqslant 0$ (if otherwise, we can replace $p(x)$ by $p(-x) = ax^2 - bx + c$). Now we substitute the values $x = 1$, $x = 0$ and $x = -1$ into the inequality $|p(x)| = |ax^2 + bx + c| \leqslant 1$, which results in

$$|a+b+c| \leqslant 1, \quad |c| \leqslant 1 \quad \text{and} \quad |a-b+c| \leqslant 1,$$
$$\text{that is } |c| \leqslant 1 \text{ and } |a+b| \leqslant 2, \quad |a-b| \leqslant 2$$

Further, if $c \geqslant 0$ then $0 \leqslant cx^2 \leqslant c$ for $|x| \leqslant 1$; for $|x| \leqslant 1$ we also have $-b \leqslant bx \leqslant b$. This means that for these values of x there hold the relations

$$p_1(x) = cx^2 + bx + a \leqslant c + b + a \leqslant 1$$
$$\text{and} \quad p_1(x) = cx^2 + bx + a \geqslant 0 + (-b) + a = a - b \geqslant -2$$

whence it follows that $|p_1(x)| \leqslant 2$. Similarly, if $c \leqslant 0$ and $c \leqslant \leqslant cx^2 \leqslant 0$ (and, as before, $-b \leqslant bx \leqslant b$) then

$$p_1(x) = cx^2 + bx + a \leqslant 0 + b + a = a + b \leqslant 2$$
$$\text{and} \quad p_1(x) = cx^2 + bx + a \geqslant c + (-b) + a = a - b + c \geqslant -1$$

whence it follows that $|p_1(x)| \leqslant 2$ in this case as well.

305. We shall exclude the case $a = 0$ which is of no interest because for $a = 0$ each of the three given equations (1), (2) and (3) is of the first degree and has a single root, all the equations coinciding (here we have $x_1 = x_2 = x_3$). We shall also exclude the case $c = 0$ when the three given equations have the roots $-b/a$ and 0, b/a and 0, $-2b/a$ and 0 respectively because in this case the root $x_3 = 0$ of equation (3) lies between any root of equation (1) and any root of equation (2).

Further, if $ax_1^2 + bx_1 + c = 0$ then

$$\frac{a}{2}x_1^2 + bx_1 + c = (ax_1^2 + bx_1 + c) - \frac{a}{2}x_1^2 = -\frac{a}{2}x_1^2$$

Similarly, if $-ax_2^2 + bx_2 + c = 0$ then

$$\frac{a}{2}x_2^2 + bx_2 + c = -ax_2^2 + bx_2 + c + \frac{3}{2}ax_2^2 = \frac{3}{2}ax_2^2$$

Consequently the expressions $(a/2)x_1^2 + bx_1 + c = -(1/2)ax_1^2$ and $(a/2)x_2^2 + bx_2 + c = (3/2)ax_2^2$ are of *different* signs. This means that the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ belonging to the parabola $y = f(x) = (a/2)x^2 + bx + c$ lie *on different sides* from the x-axis, whence it follows that between them there is an intermediate point $(x_3, 0)$ at which the parabola intersects the x-axis; the number x_3 is the sought-for root of equation (3).

306. Since α and β are the roots of the equation

$$x^2 + px + q = 0$$

we have

$$(x - \alpha)(x - \beta) = x^2 + px + q$$

Consequently

$$(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta) = [(\gamma - \alpha)(\gamma - \beta)][(\delta - \alpha)(\delta - \beta)] =$$
$$= (\gamma^2 + p\gamma + q)(\delta^2 + p\delta + q)$$

But we have

$$\gamma + \delta = -P \quad \text{and} \quad \gamma\delta = Q$$

and therefore

$$(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta)=(\gamma^2+p\gamma+q)(\delta^2+p\delta+q)=$$
$$=\gamma^2\delta^2+p\gamma^2\delta+q\gamma^2+p\gamma\delta^2+p^2\gamma\delta+pq\gamma+q\delta^2+pq\delta+q^2=$$
$$=(\gamma\delta)^2+p\gamma\delta(\gamma+\delta)+q[(\gamma+\delta)^2-2\gamma\delta]+p^2\gamma\delta+pq(\gamma+\delta)+q^2=$$
$$=Q^2-pPQ+q(P^2-2Q)+p^2Q-pqP+q^2=$$
$$=Q^2+q^2-pP(Q+q)+qP^2+p^2Q-2qQ$$

307. *First solution.* Let us find the coefficient a from the second equation and substitute it into the first equation. Then we consecutively obtain

$$a=-(x^2+x)$$
$$x^2-(x^2+x)x+1=0$$
$$x^3-1=0$$
$$(x-1)(x^2+x+1)=0$$

whence

$$x_1=1, \quad x_{2,3}=\frac{-1\pm i\sqrt{3}}{2}$$

Since $a=-(x^2+x)$, it follows that $a_1=-2$, $a_{2,3}=1$.

Second solution. Proceeding from the result of Problem 306, we can assert that for the given equations to possess at least one common root it is necessary and sufficient that the expression

$$a^2+1-a\cdot 1(a+1)+1+a^3-2a=a^3-3a+2=$$
$$=(a-1)(a^2+a-2)=(a-1)^2(a+2)$$

turn into zero.

From this relation we find

$$a_1=-2, \quad a_{2,3}=1$$

308. (a) Let $(x-a)(x-10)+1=(x+b)(x+c)$. Putting $x=-b$ in both members of this equality we obtain

$$(-b-a)(-b-10)+1=(-b+b)(-b+c)=0$$

It follows that

$$(b+a)(b+10)=-1$$

Since a and b are integers, the sums $b+a$ and $b+10$ are also integral numbers. The number -1 can be expressed as a product of two integers in only one way, namely $-1=1\cdot(-1)$, and therefore only the following two possibilities can take place here:

(1) $b+10=1$, that is $b=-9$; then $b+a=-9+a=-1$, whence $a=8$; here we have

$$(x-8)(x-10)+1=(x-9)^2$$

(2) $b+10=-1$, that is $b=-11$; then $b+a=-11+$ $+a=1$, whence $a=12$; here we have

$$(x-12)(x-10)+1=(x-11)^2$$

(b) Since a polynomial of the fourth degree can be expressed either as a product of a polynomial of the first degree by a polynomial of the third degree or as a product of two polynomials of the second degree, we have to consider separately the following two cases:

$$\text{(A)}\quad x(x-a)(x-b)(x-c)+1=(x+p)(x^3+qx^2+rx+s) \qquad (*)$$

(the coefficient in x in the first factor on the right-hand side of this equality and the coefficient in x^3 in the second factor are both equal either to 1 or to -1 because the coefficient in x^4 in the product of these factors must be equal to the coefficient in x^4 in the expression $x(x-a)(x-b)(x-c)+1$, that is to 1, and the equality $x(x-a)(x-b)(x-c)+1=(-x+p_1)(-x^3+q_1x^2+$ $r_1x+s_1)$ can be brought to form (*) by multiplying both factors on its right-hand side by -1).

On putting in succession $x=0$, $x=a$, $x=b$ and $x=c$ in equality (*) and taking into account that 1 can be factored only in the two ways $1=1\cdot 1$ and $1=(-1)\cdot(-1)$ we conclude that the four *different* numbers $0+p=p$, $a+p$, $b+p$ and $c+p$ (we remind the reader that the numbers 0, a, b and c are all different) can assume only the two values $+1$ and -1, which is impossible.

$$\text{(B)}\quad x(x-a)(x-b)(x-c)+1=(x^2+px+q)(x^2+rx+s)$$

As above, from this equality we conclude that for $x=0$, $x=a$, $x=b$ and $x=c$ both polynomials x^2+px+q and x^2+rx+s assume the value 1 or -1. Further, the quadratic trinomial x^2+px+q cannot assume one and the same value α for three distinct values of x (because, if otherwise, the quadratic equation $x^2+px+q-\alpha=0$ should have three distinct roots), whence it follows that this trinomial must assume the value 1 for some two of the four values $x=0$, $x=a$, $x=b$ and $x=c$ and the value -1 for the other two values of x. Let us suppose that $0^2+p\cdot 0+q=q=1$, and let $x=a$ be another value of x for which this trinomial takes on the same value 1. Then for $x=b$ and $x=c$ the trinomial takes on the value -1. Thus, we have

$$a^2+pa+1=1, \quad b^2+pb+1=-1, \quad c^2+pc+1=-1$$

The equality $a^2+pa=a(a+p)=0$ implies $a+p=0$, that is $p=-a$ (because, by the hypothesis, $a\neq 0$). Thus, the last two equalities take the form

$$b^2-ab=b(b-a)=-2 \quad\text{and}\quad c^2-ac=c(c-a)=-2$$

On subtracting the second of these equalities from the first we obtain

$$b^2-ab-c^2+ac=(b-c)(b+c)-a(b-c)=(b-c)(b+c-a)=0$$

whence, since $b \neq c$, it follows that $b+c-a=0$, $a=b+c$, $b-a=-c$ and $c-a=-b$. Now, from the equality

$$b(b-a)=-bc=-2$$

we find the following values of b, c and a:

$$(1)\ b=1,\quad c=2,\quad a=b+c=3$$

In this case we have

$$x(x-a)(x-b)(x-c)+1=x(x-3)(x-1)(x-2)+1=$$
$$=(x^2-3x+1)^2$$

$$(2)\ b=-1,\quad c=-2,\quad a=b+c=-3$$

In this case we have

$$x(x-a)(x-b)(x-c)+1=x(x+3)(x+1)(x+2)+1=$$
$$=(x^2+3x+1)^2$$

Similarly, if the trinomial x^2+px+q assumes the value -1 for $x=0$ and $x=a$ and the value $+1$ for $x=b$ and $x=c$, then

$$q=-1,\quad a^2+pa-1=-1,\quad b^2+pb-1=1,\quad c^2+pc-1=1$$

whence

$$p=-a,\quad b(b-a)=c(c-a)=2,\quad b^2-ab-c^2+ac=0$$

that is

$$(b-c)(b+c-a)=0,\quad a=b+c,\quad b-a=-c,\quad -bc=2$$

We thus obtain two more systems of possible values of a, b and c:

$$(3)\ b=2,\quad c=-1,\quad a=b+c=1$$

In this case we have

$$x(x-a)(x-b)(x-c)=x(x-1)(x-2)(x+1)+1=(x^2-x-1)^2$$

$$(4)\ b=1,\quad c=-2,\quad a=b+c=-1$$

In this case we have

$$x(x-a)(x-b)(x-c)=x(x+1)(x-1)(x+2)+1=(x^2+x-1)^2$$

Remark. Another solution of this problem is given at the end of the solution of Problem 309 (b).

309. **(a)** Let us suppose that

$$(x-a_1)(x-a_2)(x-a_3)\ \dots\ (x-a_n)-1=p(x)\,q(x)$$

where $p(x)$ and $q(x)$ are polynomials with integral coefficients, the sum of the degrees of $p(x)$ and $q(x)$ being equal to n. We can assume that the leading coefficients of both polynomials are equal to 1 (cf. the solution of the foregoing problem). On substituting the values $x=a_1$, $x=a_2$, $x=a_3$, $\dots$, $x=a_n$ into this equality and taking into account that -1 can be expressed as a product of integral numbers in the only way $-1=1\cdot(-1)$ we conclude that either $p(x)=1$ and $q(x)=-1$ or $p(x)=-1$ and $q(x)=1$ for each of these n values of x. Thus, the sum $p(x)+q(x)$ is equal to zero for $x_1=a_1$, $x_2=a_2$, $\dots$, $x_n=a_n$. Hence, $x_1=a_1$, $x_2=a_2$, $\dots$, $x_n=a_n$ are roots of the equation $p(x)+q(x)=0$; it follows that the polynomial $p(x)+q(x)$ is divisible by each of the binomials $x-a_1$, $x-a_2$, $\dots$, $x-a_n$, and consequently it is divisible by the product $(x-a_1)(x-a_2)\dots(x-a_n)$. Further, the degree of the equation $p(x)+q(x)=0$ coincides with the greatest of the degrees of the polynomials $p(x)$ and $q(x)$; this degree is smaller than n (n is equal to the degree of the expression $(x-a_1)(x-a_2)\dots(x-a_n)-1$). It follows that the polynomial $p(x)+q(x)$ cannot be divisible by the product $(x-a_1)\times\times(x-a_2)\dots(x-a_n)$, and consequently the factorization whose existence we have supposed is impossible.

(b) Let us suppose that

$$(x-a_1)(x-a_2)(x-a_3)\ \dots\ (x-a_n)+1=p(x)\,q(x)$$

where $p(x)$ and $q(x)$ are polynomials with integral coefficients whose leading coefficients are equal to 1. The substitution of the values $x=a_1$, $x=a_2$, $x=a_3$, $\dots$, $x=a_n$ into this equality shows that

either $\quad p(x)=1,\quad q(x)=1\quad$ or $\quad p(x)=-1,\quad q(x)=-1$

for each of these n values of x.

Thus, the difference $p(x)-q(x)$ turns into zero for n different values of x; it follows that, on the one hand, $p(x)-q(x)\equiv 0$, that is $p(x)\equiv q(x)$ (cf. the solution of Problem 309 (a)), and, on the other hand, the number n is even: $n=2k$ where k is equal to the coinciding degrees of the polynomials $p(x)$ and $q(x)$ (we have found that $p(x)\equiv q(x)$). Now let us rewrite the above equality in the form

$$(x-a_1)(x-a_2)(x-a_3)\ \dots\ (x-a_{2k})=[p(x)]^2-1$$

or, equivalently,

$$(x-a_1)(x-a_2)(x-a_3)\ \dots\ (x-a_{2k})=[p(x)+1][p(x)-1]$$

We see that the product of the two polynomials $p(x)+1$ and $p(x)-1$ turns into zero for $x=a_1$, $x=a_2$, $x=a_3$, ..., $x=a_{2k}$. Consequently, for each of these $2k$ values of x at least one of the factors on the right-hand side turns into zero. This means that $p(x)+1$ or $p(x)-1$ is divisible by $x-a_1$, $p(x)+1$ or $p(x)-1$ is divisible by $x-a_2$ etc. Since a polynomial of the kth degree cannot be divisible by a product of more than k different binomials of the form $x-a_i$ and since the divisibility of a polynomial of the kth degree with leading coefficient 1 by a product of k different binomials of the form $x-a_i$ implies that the polynomial is equal to that product, it follows that the polynomial $p(x)+1$ is equal to a product of some k of the $2k$ factors on the left-hand side of the last equality while the polynomial $p(x)-1$ is equal to the product of the other k factors.

For definiteness, let us suppose that

$$p(x)+1=(x-a_1)(x-a_3)\dots(x-a_{2k-1})$$

and

$$p(x)-1=(x-a_2)(x-a_4)\dots(x-a_{2k})$$

The subtraction of the second of these equalities from the first one yields

$$2=(x-a_1)(x-a_3)\dots(x-a_{2k-1})-(x-a_2)(x-a_4)\dots(x-a_{2k})$$

On substituting one of the values of x which we are considering, say $x=a_2$, into the last relation we arrive at a factorization of the number 2 into k integral factors:

$$2=(a_2-a_1)(a_2-a_3)\dots(a_2-a_{2k-1})$$

Since the number 2 cannot be expressed as a product of more than three different integral factors, it immediately follows that $k\leqslant 3$. It is evident that the case $k=3$ is impossible. Indeed, the number 2 can be expressed as a product of three different integral factors in only one way: $2=1\cdot(-1)\cdot(-2)$. Let us suppose that $k=3$ and that $a_1<a_3<a_5$. Then $2=(a_2-a_1)(a_2-a_3)(a_2-a_5)$ where $a_2-a_1>a_2-a_3>a_2-a_5$, and consequently $a_2-a_1=1$, $a_2-a_3=-1$ and $a_2-a_5=-2$. The substitution of $x=a_4$ into the formula

$$2=(x-a_1)(x-a_3)(x-a_5)-(x-a_2)(x-a_4)(x-a_6)$$

results in another factorization of the number 2 into three different integral factors: $2=(a_4-a_1)(a_4-a_3)\ (a_4-a_5)$ where again $a_4-a_1>a_4-a_3>a_4-a_5$. It follows that $a_4-a_1=1$, $a_4-a_3=-1$ and $a_4-a_5=-2$, and hence $a_4=a_2$, which contradicts the condition of the problem.

Thus, there are only two possible cases here: $k=2$ and $k=1$. Let us consider them separately.

1°. If $k = 1$ then

$$2 = (x - a_1) - (x - a_2)$$

whence $a_2 = a_1 + 2$. Denoting a_1 simply as a we obtain

$$(x - a_1)(x - a_2) + 1 = (x - a)(x - a - 2) + 1 = (x - a - 1)^2$$

(cf. the solution of Problem 308 (a)).

2°. If $k = 2$ then

$$2 = (x - a_1)(x - a_3) - (x - a_2)(x - a_4)$$

For definiteness, let $a_1 < a_3$ and $a_2 < a_4$. The substitution of $x = a_2$ and $x = a_4$ into the last equality results in

$$2 = (a_2 - a_1)(a_2 - a_3), \quad a_2 - a_1 > a_2 - a_3$$

and

$$2 = (a_4 - a_1)(a_4 - a_3), \quad a_4 - a_1 > a_4 - a_3$$

The number 2 can be expressed as a product of two factors written in a decreasing order only in two ways: $2 = 2 \cdot 1$ or $2 = (-1) \cdot (-2)$. Besides, we have $a_2 - a_1 < a_4 - a_1$, and therefore

$$a_2 - a_1 = -1, \quad a_2 - a_3 = -2$$

and

$$a_4 - a_1 = 2, \quad a_4 - a_3 = 1$$

Now, denoting a_1 as a, we obtain

$$a_2 = a - 1, \quad a_3 = a + 1, \quad a_4 = a + 2$$

and

$$(x - a_1)(x - a_2)(x - a_3)(x - a_4) + 1 =$$
$$= (x - a)(x - a + 1)(x - a - 1)(x - a - 2) + 1 =$$
$$= [x^2 - (2a - 1)x + a^2 + a - 1]^2$$

(cf. the solution of Problem 308 (b)).

310. By analogy with the solution of the foregoing problem, we conclude that from the equality

$$(x - a_1)^2 (x - a_2)^2 (x - a_3)^2 \ldots (x - a_n)^2 + 1 = p(x)\,q(x) \qquad (*)$$

where $p(x)$ and $q(x)$ are some polynomials with integral coefficients (whose leading coefficients are equal to 1) it follows that either $p(x) = 1$ and $q(x) = 1$ or $p(x) = -1$ and $q(x) = -1$ for each of the values $x = a_1$, $x = a_2$, $x = a_3$, ..., $x = a_n$. Let us show that the polynomial $p(x)$ and also the polynomial $q(x)$ are either equal to 1 for all the values $x = a_1$, $x = a_2$, ..., $x = a_n$ or are equal to -1 for all these values of x.

Indeed, if, for instance, the polynomial $p(x)$ took on the value 1 for $x = a_i$ and the value -1 for $x = a_j$ then it would turn into zero for an intermediate value of x lying between a_i and a_j (if the point of the graph of the function $y = p(x)$ corresponding to $x = a_i$ lies above the x-axis and the point of the graph corresponding to $x = a_j$ lies below that axis then the continuous curve $y = p(x)$ must intersect the x-axis at a point lying somewhere between $x = a_i$ and $x = a_j$), which is impossible because the left-hand side of equality (*) is always greater than or equal to 1 and therefore it cannot turn into zero.

Now let us suppose that both $p(x)$ and $q(x)$ assume the value 1 for $x = a_1$, $x = a_2$, ..., $x = a_n$. In this case both $p(x) - 1$, $q(x) - 1$ turn into zero for $x = a_1$, $x = a_2$, ..., $x = a_n$, and consequently $p(x) - 1$ and $q(x) - 1$ are divisible by the product $(x - a_1)(x - a_2) \dots (x - a_n)$. Since the sum of the degrees of the polynomials $p(x)$ and $q(x)$ is equal to the degree of the expression $(x - a_1)^2 (x - a_2)^2 \dots (x - a_n)^2 + 1$, that is to $2n$, we have $p(x) - 1 = (x - a_1) \dots (x - a_n)$ and $q(x) - 1 = (x - a_1) \dots$ $\dots (x - a_n)$ (cf. the solution of the foregoing problem).

Thus, we arrive at the equality

$$(x - a_1)^2 (x - a_2)^2 \dots (x - a_n)^2 + 1 = p(x)\,q(x) =$$
$$= [(x - a_1) \dots (x - a_n) + 1][(x - a_1) \dots (x - a_n) + 1] =$$
$$= (x - a_1)^2 (x - a_2)^2 \dots (x - a_n)^2 + 2(x - a_1)(x - a_2) \dots (x - a_n) + 1$$

whence follows the equality

$$(x - a_1)(x - a_2) \dots (x - a_n) \equiv 0$$

which is impossible. In the same way we can prove that $p(x)$ and $q(x)$ cannot simultaneously assume the value -1 at the points $x = a_1$, $x = a_2$, ..., $x = a_n$ (in this case the assumption that $p(x) = q(x) = -1$ for $x = a_1$, $x = a_2$, ..., $x = a_n$ would imply $(x - a_1)(x - a_2) \dots (x - a_n) - 1 = p(x) = q(x)$).

Thus, we see that the expression

$$(x - a_1)^2 (x - a_2)^2 \dots (x - a_n)^2 + 1$$

cannot be expressed as a product of two polynomials with integral coefficients.

311. Let the polynomial $P(x)$ take on the value 7 at the points $x = a$, $x = b$, $x = c$ and $x = d$. Then a, b, c and d are four integral roots of the equation $P(x) - 7 = 0$. This means that the polynomial $P(x) - 7$ is divisible by $x - a$, $x - b$, $x - c$ and

$x-d$ *, that is

$$P(x)-7=(x-a)(x-b)(x-c)(x-d)p(x)$$

where $p(x)$ may be equal to 1.

Now let us suppose that the polynomial $P(x)$ assumes the value 14 for an integral value $x=A$. On substituting $x=A$ into the last equality we obtain

$$7=(A-a)(A-b)(A-c)(A-d)p(A)$$

which is impossible because the integral numbers $A-a$, $A-b$, $A-c$ and $A-d$ are all distinct and the number 7 cannot be factored into five integers among which at least four are different.

312. If a polynomial $P(x)$ of the seventh degree is expressed as a product of two polynomials $p(x)$ and $q(x)$ with integral coefficients then the degree of at least one of the factors $p(x)$ and $q(x)$ does not exceed 3; let us suppose $p(x)$ is that factor of a degree not higher than 3. If $P(x)$ assumes the values ± 1 for seven integral values of x then for the same values of x the polynomial $p(x)$ also assumes the values ± 1 (because $p(x)q(x)=P(x)$). Among the seven integral values of x for which $p(x)$ assumes the values ± 1 there are four values for which $p(x)$ is equal to 1 or four values for which $p(x)$ is equal to -1. In the first case the third-degree equation $p(x)-1=0$ possesses four roots and in the second case the equation $p(x)+1=0$ possesses four roots. Neither of these cases can take place; for instance, in the first case the polynomial $p(x)=1$ must be divisible by a polynomial of the fourth degree (cf. the solution of Problem 309 (a)); this contradiction proves the assertion of the problem.

313. Let p and q be two integral numbers simultaneously even or odd. Then the difference $P(p)-P(q)$ is even. Indeed, the expression

$$P(p)-P(q)=a_0(p^n-q^n)+a_1(p^{n-1}-q^{n-1})+\dots$$
$$\dots+a_{n-2}(p^2-q^2)+a_{n-1}(p-q)$$

is divisible by the even number $p-q$.

In particular, the difference $P(p)-P(0)$ is even for an even p. By the condition of the problem, the number $P(0)$ is odd, and consequently $P(p)$ is also odd; therefore $P(p)\neq 0$. Similarly, for an odd p the difference $P(p)-P(1)$ is even; since by the condi-

* If we suppose that the division of $P(x)-7$ by $x-a$ leaves a remainder r, that is

$$P(x)-7=(x-a)Q(x)+r$$

then the substitution of $x=a$ into this equality results in $7-7=0+r$, whence $r=0$, and consequently $P(x)-7$ is equal to $(x-a)Q(x)$ and is divisible by $(x-a)$.

tion of the problem, $P(1)$ is odd, it follows, as above, that $P(p) \neq 0$.

Consequently, $P(x)$ cannot turn into zero for any integral (even or odd) value of x, that is the polynomial $P(x)$ possesses no integral roots.

314. Let us suppose that the equation $P(x) = 0$ possesses a rational root $x = k/l$: $P(k/l) = 0$. Let us expand the polynomial $P(x)$ in powers of $x - p$, that is let us write it in the form

$$P(x) = c_0(x-p)^n + c_1(x-p)^{n-1} + \\ + c_2(x-p)^{n-2} + \ldots + c_{n-1}(x-p) + c_n$$

where $c_0, c_1, c_2, \ldots, c_n$ are some integral numbers which can easily be found when the coefficients $a_0, a_1, \ldots, a_n$ are known (the number c_0 is equal to the leading coefficient a_0 of the polynomial $P(x)$, the number c_1 is equal to the leading coefficient of the polynomial $P(x) - c_0(x-p)^n$ of the $(n-1)$th degree, the number c_2 is equal to the leading coefficient of the polynomial $P(x) - c_0(x-p)^n - c_1(x-p)^{n-1}$ of the $(n-2)$th degree etc.). On substituting $x = p$ into the last expression of $P(x)$ we obtain $c_n = P(p) = \pm 1$.

The substitution of $x = k/l$ into the same expression and the multiplication of the result by l^n yields

$$l^n P\left(\frac{k}{l}\right) = c_0(k-pl)^n + c_1 l(k-pl)^{n-1} + \\ + c_2 l^2 (k-pl)^{n-2} + \ldots + c_{n-1} l^{n-1}(k-pl) + c_n l^n = 0$$

whence it follows that if $P(k/l) = 0$ then the expression

$$\frac{c_n l^n}{k-pl} = \frac{\pm l^n}{k-pl} = -c_0(k-pl)^{n-1} - \\ - c_1 l(k-pl)^{n-2} - \ldots - c_{n-2} l^{n-2}(k-pl) - c_{n-1} l^{n-1}$$

is an integral number. Since pl is divisible by l and k is relatively prime to l (because k/l is an irreducible fraction), the number $k - pl$ is relatively prime to l, and consequently $k - pl$ is relatively prime to l^n as well. It follows that $\pm l^n/(k-pl)$ can be an integral number only when $k - pl = \pm 1$.

In just the same way we can also prove that $k - ql = \pm 1$.

Now we subtract the equality $k - pl = \pm 1$ from the equality $k - ql = \pm 1$ and obtain

$$(p-q)l = 0 \quad \text{or} \quad (p-q)l = \pm 2$$

Further, $(p-q)l > 0$ because $p > q$ and $l > 0$, and consequently $(p-q)l = 2$, $k - pl = -1$ and $k - ql = 1$.

Thus, if $p - q > 2$, the equation $P(x) = 0$ cannot have rational roots. In case $p - q = 2$ or $p - q = 1$ a rational root k/l may exist. In this case the addition of the equalities

$$k - pl = -1 \quad \text{and} \quad k - ql = 1$$

yields

$$2k - (p + q)l = 0 \quad \text{whence} \quad \frac{k}{l} = \frac{p + q}{2}$$

which is what we had to prove.

315. (a) Let us suppose that the given polynomial can be expressed as a product of two polynomials with integral coefficients:

$$x^{2222} + 2x^{2220} + 4x^{2218} + \dots + 2220x^2 + 2222 =$$
$$= (a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0)(b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \dots + b_0)$$

where $m + n = 2222$. Then $a_0 b_0 = 2222$, and consequently one of the two integral numbers a_0 and b_0 is even and the other is odd. Let us suppose that a_0 is an even number and b_0 is an odd number. We must show that in this case all the coefficients of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ must be even. Let a_k be the first (counting from right to left) odd coefficient of that polynomial. After the parentheses are opened in the product

$$(a_n x^n + a_{n-1}x^{n-1} + \dots + a_0)(b_m x^m + b_{m-1}x^{m-1} + \dots + b_0)$$

the coefficient in x^k is equal to

$$a_k b_0 + a_{k-1}b_1 + a_{k-2}b_2 + \dots + a_0 b_k \qquad (*)$$

(in case $k > m$ this sum ends with the term $a_{k-m}b_m$). This coefficient is equal to the corresponding coefficient in x^k of the original polynomial, that is, it is equal to zero when k is odd and is an even number when k is even (because all the coefficients of the polynomial indicated in the condition of the problem, except the first one, are even and $k \leqslant n < 2222$). By the hypothesis, all the numbers $a_{k-1}, a_{k-2}, a_{k-3}, \dots, a_0$ are even, and, consequently, in sum (*), all the terms except the first one, are even; therefore the product $a_k b_0$ must also be even, which is impossible since the numbers a_k and b_0 are odd.

Thus, we see that all the coefficients of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ must be even, which contradicts the fact that the product $a_n b_m$ is equal to unity. Consequently, the assumption that the given polynomial can be expressed as a product of two polynomials with integral coefficients is false.

(b) Let us put $x = y + 1$. Then we have

$$x^{250} + x^{249} + x^{248} + \dots + x + 1 =$$
$$= (y+1)^{250} + (y+1)^{249} + \dots + (y+1) + 1 =$$
$$= \frac{(y+1)^{251} - 1}{(y+1) - 1} = \frac{1}{y}[(y+1)^{251} - 1] =$$
$$= y^{250} + 251y^{249} + C(251, 2)\,y^{248} + C(251, 3)\,y^{247} + \dots$$
$$\dots + C(251, 2)\,y + 251$$

Further, taking into account that all the coefficients of the resultant polynomial, except the first one, are divisible by the prime number 251 (because $C(251, k) = [251\cdot 250\cdot 249 \dots (251-k+1)]/(1\cdot 2\cdot 3 \dots k)$) and that the constant term of the polynomial is equal to 251 and is not divisible by 251^2, we can repeat almost literally the argument used in the solution of Problem 315 (a) (the only distinction is that instead of the divisibility of the coefficients by 2 we should analyze the divisibility by 251). In this way we prove that if the given polynomial could be factored into two polynomials with integral coefficients then all the coefficients of one of the polynomials would be divisible by 251, which is impossible because the leading coefficient of the original polynomial is equal to 1.

316. Let us write the given polynomials as

$$A = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

and

$$B = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$$

By the condition of the problem, not all coefficients of the product AB are divisible by 4, and therefore it is impossible for all the coefficients of the two polynomials to be even. Consequently, some coefficients of at least one of them, say of B, are odd. Let us suppose that some of the coefficients of the polynomial A are also odd. We shall consider the first of such coefficients (the one having the smallest index); let a_s be that coefficient. Further, let the first odd coefficient of the polynomial B be b_k. We shall consider the coefficient in x^{k+s} in the product of the polynomials A and B. The term x^{k+s} in the product is formed of the products of those powers of x in A and B the sum of whose exponents is equal to $k+s$. Consequently, this coefficient is equal to

$$a_0b_{k+s} + a_1b_{k+s-1} + \dots + a_{s-1}b_{k+1} + a_sb_k + a_{s+1}b_{k-1} + \dots + a_{s+k}b_0.$$

All products in this sum which precede the term a_sb_k are even because such are the numbers $a_0, a_1, \dots, a_{s-1}$. All products following a_sb_k are also even because $b_{k-1}, b_{k-2}, \dots, b_0$ are even numbers. As to the product a_sb_k, it is an odd number because such are the numbers a_s and b_k. Consequently, the whole sum is odd, which contradicts the fact that all coefficients in the product are even. We see that the assumption that some of the coefficients of the polynomial A are odd is false, and therefore all coefficients of A are even numbers, which is what we intended to prove.

317. Let us prove that for any rational nonintegral value of x the polynomial $P(x)$ cannot assume an integral value, and hence it cannot be equal to zero since zero is an integral number.

Let $x = p/q$ where p and q are relatively prime. Then

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n =$$
$$= \frac{p^n}{q^n} + a_1\frac{p^{n-1}}{q^{n-1}} + a_2\frac{p^{n-2}}{q^{n-2}} + \dots + a_{n-1}\frac{p}{q} + a_n =$$
$$= \frac{p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \dots + a_{n-1}pq^{n-1} + a_nq^n}{q^n} =$$
$$= \frac{p^n + q(a_1p^{n-1} + a_2p^{n-2}q + \dots + a_{n-1}pq^{n-2} + a_nq^{n-1})}{q^n}$$

The number p^n, like the number p, is relatively prime to q; consequently, the number $p^n + q(a_1p^{n-1} + \dots + a_nq^{n-1})$ is also relatively prime to q and hence to q^n as well. Therefore the rightmost fraction in the last relation is irreducible and therefore it cannot be equal to an integral number.

318. Let N be an integral number and let $P(N) = M$. The expression

$$P(N + kM) - P(N) = a_0[(N + kM)^n - N^n] +$$
$$+ a_1[(N + kM)^{n-1} - N^{n-1}] + \dots + a_{n-1}[(N + kM) - N]$$

is divisible by kM for any integral k (because $(N + kM)^l - N^l$ is divisible by $(N + kM) - N = kM$) and hence by M as well; consequently, $P(N + kM)$ is divisible by M for any integral k.

Therefore if we prove that among the values $P(N + kM)$ $(k = 0, 1, 2, \dots)$ there are numbers different from $\pm M$, this will imply that not all these values are prime numbers. To prove what has been said we take into account that for any A the polynomial $P(x)$ of the nth degree assumes the value equal to A for not more than n different values of x (because the equation $P(x) - A = 0$ of the nth degree cannot have more than n roots). Thus, among the $2n + 1$ first values $P(N + kM)$ $(k = 0, 1, 2, \dots, 2n)$ there is at least one different from M and from $-M$.

319. First of all it should be noted that every polynomial $P(x)$ of the nth degree can be represented as a linear combination of the polynomials

$$P_0(x) = 1, \quad P_1(x) = x,$$
$$P_2(x) = \frac{x(x-1)}{1 \cdot 2}, \dots, P_n(x) = \frac{x(x-1)(x-2)\dots(x-n+1)}{1 \cdot 2 \cdot 3 \dots n}$$

with some coefficients, that is

$$P(x) = b_nP_n(x) + b_{n-1}P_{n-1}(x) + \dots + b_1P_1(x) + b_0P_0(x)$$

To prove this property we take into account that if the number b_n is such that $b_n/n!$ is equal to the leading coefficient of the polynomial $P(x)$ then $P(x)$ and $b_nP_n(x)$ have equal coefficients in x^n, if b_{n-1} is such that $b_{n-1}/(n-1)!$ is equal to the leading coefficient of $P(x) - b_nP_n(x)$ then the coefficient in x^n and the coefficient in x^{n-1} of the polynomial $P(x)$ coincide with those of the polynomial $b_nP_n(x) + b_{n-1}P_{n-1}(x)$, if b_{n-2} is such that $b_{n-2}/(n-2)!$ is equal to the leading coefficient of the polynomial $P(x) - b_nP_n(x) - b_{n-1}P_{n-1}(x)$ then $P(x)$ and $b_nP_n(x) + b_{n-1}P_{n-1}(x) + b_{n-2}P_{n-2}(x)$ have the same coefficients in x^n, in x^{n-1} and in x^{n-2} etc. This means that the coefficients $b_n, b_{n-1}, \dots, b_1, b_0$ can be chosen so that the polynomials $P(x)$ and $b_nP_n(x) + b_{n-1}P_{n-1}(x) + \dots + b_1P_1(x) + b_0P_0(x)$ coincide completely.

Now let $P(x)$ be a polynomial of the nth degree such that $P(0), P(1), \dots, P(n)$ are integral numbers. According to what has been proved, this polynomial can be represented in the form

$$P(x) = b_0P_0(x) + b_1P_1(x) + b_2P_2(x) + \dots + b_nP_n(x)$$

Now we note that

$$P_1(0) = P_2(0) = \dots = P_n(0) = \\ = P_2(1) = P_3(1) = \dots = P_n(1) = P_3(2) = \dots = P_n(2) = \dots \\ \dots = P_{n-1}(n-2) = P_n(n-2) = P_n(n-1) = 0$$

and

$$P_0(0) = P_1(1) = P_2(2) = \dots = P_{n-1}(n-1) = P_n(n) = 1$$

Therefore

$$P(0) = b_0P_0(0) \quad \text{whence} \quad b_0 = P(0)$$

$$P(1) = b_0P_0(1) + b_1P_1(1) \quad \text{whence} \quad b_1 = P(1) - b_0P_0(1)$$

$$P(2) = b_0P_0(2) + b_1P_1(2) + b_2P_2(2)$$

whence $b_2 = P(2) - b_0P_0(2) - b_1P_1(2)$

. . . .

$$P(n) = b_0P_0(n) + b_1P_1(n) + \dots + b_{n-1}P_{n-1}(n) + b_nP_n(n)$$

whence

$$b_n = P(n) - b_0P_0(n) - b_1P_1(n) - \dots - b_{n-1}P_{n-1}(n)$$

Thus, all the coefficients $b_0, b_1, b_2, \dots, b_n$ are integral numbers.

320. (a) From the solution of Problem 319 it follows that a polynomial of the indicated kind can be written as a linear combination of the polynomials $P_0(x)$, $P_1(x)$, ..., $P_n(x)$ with *integral* coefficients. This property and the fact that the polynomials $P_0(x)$, $P_1(x)$, ..., $P_n(x)$ take on integral values for *every* integral x (see Problem 75 (a)) imply the assertion stated in the condition of the problem.

(b) If a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ assumes integral values for $x = k, k+1, k+2, \ldots, k+n$ then the polynomial $Q(x) = P(x+k) = a_n(x+k)^n + a_{n-1}(x+k)^{n-1} + \ldots + a_1(x+k) + a_0$ assumes integral values for $x = 0, 1, 2, 3, \ldots, n$. By virtue of the solution of Problem 320 (a), it follows that $Q(x)$ assumes integral values for all integral x, whence we conclude that the polynomial $P(x) = Q(x-k)$ also assumes integral values for all integral x.

(c) Let a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ assume integral values for $x = 0, 1, 4, 9, \ldots, n^2$. Then the polynomial $Q(x) = P(x^2) = a_n(x^2)^n + a_{n-1}(x^2)^{n-1} + \ldots + a_1 x^2 + a_0$ of the $2n$th degree assumes integral values for the $2n+1$ consecutive integral values $x = -n, -(n-1), -(n-2), \ldots, -1, 0, 1, \ldots, n-1, n$. Indeed, we obviously have $Q(0) = P(0)$, $Q(1) = Q(-1) = P(1)$, $Q(2) = Q(-2) = P(4)$, $Q(3) = Q(-3) = P(9)$, ..., $Q(n) = Q(-n) = P(n^2)$ and, according to the condition of the problem, all these numbers are integral. Consequently, by virtue of the solution of Problem 320 (b), the polynomial $Q(x)$ assumes integral values for all integral values of x. This means that the expression $P(k^2) = Q(k)$ is an integral number for any integral k.

As an example, we can take the polynomial $P(x) = x(x-1)/12$ for which

$$Q(x) = P(x^2) = \frac{x^2(x^2-1)}{12} = \frac{x^2(x-1)(x+1)}{12} =$$
$$= 2\frac{(x+2)(x+1)x(x-1)}{1\cdot 2\cdot 3\cdot 4} - \frac{(x+1)x(x-1)}{1\cdot 2\cdot 3}$$

321. (a) Using De Moivre's formula and Newton's binomial formula we write

$$\cos 5\alpha + i\sin 5\alpha = (\cos\alpha + i\sin\alpha)^5 =$$
$$= \cos^5\alpha + 5\cos^4\alpha i\sin\alpha + 10\cos^3\alpha(i\sin\alpha)^2 +$$
$$+ 10\cos^2\alpha(i\sin\alpha)^3 + 5\cos\alpha(i\sin\alpha)^4 + (i\sin\alpha)^5 =$$
$$= (\cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha) +$$
$$+ i(5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha)$$

On equating the real and the imaginary parts on the left-hand and right-hand sides we derive the required formulas.

(b) By analogy with the solution of Problem 321 (a), we have

$$\cos n\alpha + i \sin n\alpha = (\cos\alpha + i\sin\alpha)^n =$$
$$= \cos^n\alpha + C(n,\ 1)\cos^{n-1}\alpha i \sin\alpha + C(n,\ 2)\cos^{n-2}\alpha (i\sin\alpha)^2 +$$
$$+ C(n,\ 3)\cos^{n-3}\alpha (i\sin\alpha)^3 + C(n,\ 4)\cos^{n-4}\alpha (i\sin\alpha)^4 + \ldots =$$
$$= (\cos^n\alpha - C(n,\ 2)\cos^{n-2}\alpha\sin^2\alpha + C(n,\ 4)\cos^{n-4}\alpha\sin^4\alpha - \ldots) +$$
$$+ i(C(n,\ 1)\cos^{n-1}\alpha\sin\alpha - C(n,\ 3)\cos^{n-3}\alpha\sin^3\alpha + \ldots)$$

whence follow the required formulas.

322. According to the formulas derived in the solution of Problem 321 (b), we have

$$\tan 6\alpha = \frac{\sin 6\alpha}{\cos 6\alpha} = \frac{6\cos^5\alpha\sin\alpha - 20\cos^3\alpha\sin^3\alpha + 6\cos\alpha\sin^5\alpha}{\cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha}$$

The division of the numerator and the denominator of the last fraction by $\cos^6\alpha$ yields the required formula:

$$\tan 6\alpha = \frac{6\tan\alpha - 20\tan^3\alpha + 6\tan^5\alpha}{1 - 15\tan^2\alpha + 15\tan^4\alpha - \tan^6\alpha}$$

323. Let us rewrite the equation $x + 1/x = 2\cos\alpha$ in the form

$$x^2 + 1 = 2x\cos\alpha$$

that is

$$x^2 - 2x\cos\alpha + 1 = 0$$

It follows that

$$x = \cos\alpha \pm \sqrt{\cos^2\alpha - 1} = \cos\alpha \pm i\sin\alpha$$

whence we find

$$x^n = \cos n\alpha \pm i\sin n\alpha$$

and

$$\frac{1}{x^n} = \frac{1}{\cos n\alpha \pm i\sin n\alpha} = \cos n\alpha \mp i\sin n\alpha$$

On performing the addition we obtain

$$x^n + \frac{1}{x^n} = 2\cos n\alpha$$

324. Let us consider the sum

$$[\cos\varphi + i\sin\varphi] + [\cos(\varphi+\alpha) + i\sin(\varphi+\alpha)] +$$
$$+ [\cos(\varphi+2\alpha) + i\sin(\varphi+2\alpha)] + \ldots + [\cos(\varphi+n\alpha) + i\sin(\varphi+n\alpha)]$$

The problem reduces to the computation of the real and the imaginary part of this sum. Denoting $\cos\varphi + i\sin\varphi$ as a and $\cos\alpha + i\sin\alpha$ as x and using the formula for the multiplication of complex numbers and De Moivre's formula we find that this sum is equal to

$$a+ax+ax^2+\ \dots\ +ax^n=\frac{ax^{n+1}-a}{x-1}=$$

$$=(\cos\varphi+i\sin\varphi)\frac{\cos(n+1)\alpha+i\sin(n+1)\alpha-1}{\cos\alpha+i\sin\alpha-1}=$$

$$=(\cos\varphi+i\sin\varphi)\frac{[(\cos(n+1)\alpha-1)+i\sin(n+1)\alpha]}{[(\cos\alpha-1)+i\sin\alpha]}=$$

$$=(\cos\varphi+i\sin\varphi)\frac{-2\sin^2\frac{n+1}{2}\alpha+2i\sin\frac{n+1}{2}\alpha\cos\frac{n+1}{2}\alpha}{-2\sin^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=$$

$$=(\cos\varphi+i\sin\varphi)\frac{2i\sin\frac{n+1}{2}\alpha\left[\cos\frac{n+1}{2}\alpha+i\sin\frac{n+1}{2}\alpha\right]}{2i\sin\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]}=$$

$$=\frac{\sin\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}(\cos\varphi+i\sin\varphi)\times$$

$$\times\frac{\left(\cos\frac{n+1}{2}\alpha+i\sin\frac{n+1}{2}\alpha\right)\left(\cos\frac{\alpha}{2}-i\sin\frac{\alpha}{2}\right)}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}=$$

$$=\frac{\sin\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}\left[\cos\left(\varphi+\frac{n}{2}\alpha\right)+i\sin\left(\varphi+\frac{n}{2}\alpha\right)\right]$$

(in the last transformation we have again used the multiplication formula for complex numbers and the fact that $\cos\alpha/2-$ $-i\sin\alpha/2=\cos(-\alpha/2)+i\sin(-\alpha/2)$). From this expression readily follow the required formulas.

325. Using the formula $\cos^2 x=(1+\cos 2x)/2$ and the result of the foregoing problem we derive

$$\cos^2\alpha+\cos^2 2\alpha+\ \dots\ +\cos^2 n\alpha=$$

$$=\frac{1}{2}[\cos 2\alpha+\cos 4\alpha+\ \dots\ +\cos 2n\alpha+n]=$$

$$=\frac{1}{2}\left[\frac{\sin(n+1)\alpha\cos n\alpha}{\sin\alpha}-1\right]+\frac{n}{2}=\frac{\sin(n+1)\alpha\cos n\alpha}{2\sin\alpha}+\frac{n-1}{2}$$

Since $\sin^2 x=1-\cos^2 x$, it follows that

$$\sin^2\alpha+\sin^2 2\alpha+\ \dots\ +\sin^2 n\alpha=$$

$$=n-\frac{\sin(n+1)\alpha\cos n\alpha}{2\sin\alpha}-\frac{n-1}{2}=\frac{n+1}{2}-\frac{\sin(n+1)\alpha\cos n\alpha}{2\sin\alpha}$$

326. It is required to compute the real and the imaginary part of the sum

$$(\cos\alpha + i\sin\alpha) + C(n, 1)(\cos 2\alpha + i\sin 2\alpha) + \\ + C(n, 2)(\cos 3\alpha + i\sin 3\alpha) + \dots + (\cos(n+1)\alpha + i\sin(n+1)\alpha)$$

Let us denote $\cos\alpha + i\sin\alpha$ by x; the application of De Moivre's formula and Newton's binomial formula makes it possible to transform this sum as follows:

$$x + C(n, 1)x^2 + C(n, 2)x^3 + \dots + x^{n+1} = x(x+1)^n =$$
$$= (\cos\alpha + i\sin\alpha)(\cos\alpha + 1 + i\sin\alpha)^n =$$
$$= (\cos\alpha + i\sin\alpha)\left(2\cos^2\frac{\alpha}{2} + 2i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)^n =$$
$$= 2^n\cos^n\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)\left(\cos\frac{n\alpha}{2} + i\sin\frac{n\alpha}{2}\right) =$$
$$= 2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{n+2}{2}\alpha + i\sin\frac{n+2}{2}\alpha\right)$$

From the last expression we conclude that

$$\cos\alpha + C(n, 1)\cos 2\alpha + C(n, 2)\cos 3\alpha + \dots \\ \dots + \cos(n+1)\alpha = 2^n\cos^n\frac{\alpha}{2}\cos\frac{n+2}{2}\alpha$$

and

$$\sin\alpha + C(n, 1)\sin 2\alpha + C(n, 2)\sin 3\alpha + \dots \\ \dots + \sin(n+1)\alpha = 2^n\cos^n\frac{\alpha}{2}\sin\frac{n+2}{2}\alpha$$

327. We shall make use of the formula

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

which makes it possible to write the given sum in the form

$$\frac{1}{2}\left[\cos\frac{(m-n)\pi}{p} + \cos\frac{2(m-n)\pi}{p} + \cos\frac{3(m-n)\pi}{p} + \dots \right. \\ \left. \dots + \cos\frac{(p-1)(m-n)\pi}{p}\right] - \frac{1}{2}\left[\cos\frac{(m+n)\pi}{p} + \right. \\ \left. + \cos\frac{2(m+n)\pi}{p} + \cos\frac{3(m+n)\pi}{p} + \dots + \cos\frac{(p-1)(m+n)\pi}{p}\right]$$

Further, the sum

$$\cos\frac{k\pi}{p} + \cos\frac{2k\pi}{p} + \cos\frac{3k\pi}{p} + \dots + \cos\frac{(p-1)k\pi}{p}$$

is equal to $p-1$ if k is divisible by $2p$ (in this case every term of the sum is equal to 1). If k is not divisible by $2p$, then, by virtue of the result of Problem 324, this sum is equal to

$$\frac{\sin\frac{pk\pi}{2p}\cos\frac{(p-1)k\pi}{2p}}{\sin\frac{k\pi}{2p}}-1=\sin k\frac{\pi}{2}\cdot\frac{\cos\left(k\frac{\pi}{2}-\frac{k\pi}{2p}\right)}{\sin\frac{k\pi}{2p}}-1=$$

$$=\begin{cases}0 & \text{for odd } k\\ -1 & \text{for even } k\end{cases}$$

It should also be noted that both numbers $m+n$ and $m-n$ are simultaneously even or odd; in particular, if $m+n$ and $m-n$ are divisible by $2p$ then both these numbers are even, whence follows the equality indicated in the condition of the problem.

328. Let us consider the equation $x^{2n+1}-1=0$; its roots are

$$1,\quad \cos\frac{2\pi}{2n+1}+i\sin\frac{2\pi}{2n+1},\quad \cos\frac{4\pi}{2n+1}+i\sin\frac{4\pi}{2n+1},\ \ldots$$

$$\ldots,\ \cos\frac{4n\pi}{2n+1}+i\sin\frac{4n\pi}{2n+1}$$

Since the coefficient in x^{2n} in the equation is equal to zero, the sum of all these roots is equal to zero:

$$\left[1+\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\ \ldots\ +\cos\frac{4n\pi}{2n+1}\right]+$$

$$+i\left[\sin\frac{2\pi}{2n+1}+\sin\frac{4\pi}{2n+1}+\ \ldots\ +\sin\frac{4n\pi}{2n+1}\right]=0$$

Consequently, each of the expressions in the brackets is equal to zero, whence, in particular,

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\ \ldots\ +\cos\frac{4n\pi}{2n+1}=-1$$

Further, we have

$$\cos\frac{2\pi}{2n+1}=\cos\frac{4n\pi}{2n+1},\qquad \cos\frac{4\pi}{2n+1}=\cos\frac{(4n-2)\pi}{2n+1}$$

etc., and hence

$$2\left[\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\ \ldots\ +\cos\frac{2n\pi}{2n+1}\right]=-1$$

that is

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\ \ldots\ +\cos\frac{2n\pi}{2n+1}=-\frac{1}{2}$$

Remark. This problem can also be solved on the basis of the formulas of Problem 324.

329. **(a)** By virtue of the result of Problem 321 (b), we have

$$\sin(2n+1)\alpha = C(2n+1,\ 1)(1-\sin^2\alpha)^n\sin\alpha - $$
$$- C(2n+1,3)(1-\sin^2\alpha)^{n-1}\sin^3\alpha + \ldots + (-1)^n\sin^{2n+1}\alpha$$

It follows that the numbers 0, $\sin\pi/(2n+1)$, $\sin 2\pi/(2n+1), \ldots$ $\ldots, \sin n\pi/(2n+1)$,

$$\sin\left(-\frac{\pi}{2n+1}\right) = -\sin\frac{\pi}{2n+1},\ \sin\left(-\frac{2\pi}{2n+1}\right) =$$
$$= -\sin\frac{2\pi}{2n+1}, \ldots, \sin\left(-\frac{n\pi}{2n+1}\right) =$$
$$= -\sin\frac{n\pi}{2n+1}$$

are the roots of the equation

$$C(2n+1,\ 1)(1-x^2)^n x - C(2n+1,\ 3)(1-x^2)^{n-1}x^3 + \ldots$$
$$\ldots + (-1)^n x^{2n+1} = 0$$

of the $(2n+1)$th degree.

Consequently, the numbers $\sin^2\pi/(2n+1)$, $\sin^2 2\pi/(2n+1), \ldots$ $\ldots, \sin^2 n\pi/(2n+1)$ are the roots of the equation

$$C(2n+1,1)(1-x)^n - C(2n+1,3)(1-x)^{n-1}x + \ldots + (-1)^n x^n = 0$$

of the nth degree.

(b) Let us replace n by $2n+1$ in the formula established in the solution of Problem 321 (b) and write this formula in the form

$$\sin(2n+1)\alpha = \sin^{2n+1}\alpha\,(C(2n+1,1)\cot^{2n}\alpha - C(2n+1,3)\cot^{2n-2}\alpha +$$
$$+ C(2n+1,5)\cot^{2n-4}\alpha - \ldots)$$

It follows that for $\alpha = \pi/(2n+1)$, $2\pi/(2n+1)$, $3\pi/(2n+1), \ldots$ $\ldots, n\pi/(2n+1)$ there holds the equality

$$C(2n+1,1)\cot^{2n}\alpha - C(2n+1,3)\cot^{2n-2}\alpha +$$
$$+ C(2n+1,5)\cot^{2n-4}\alpha - \ldots = 0$$

It follows that the numbers $\cot^2\pi/(2n+1)$, $\cot^2 2\pi/(2n+1), \ldots$ $\ldots, \cot^2 n\pi/(2\pi+1)$ are the roots of the equation

$$C(2n+1,1)x^n - C(2n+1,3)x^{n-1} + C(2n+1,5)x^{n-2} - \ldots = 0$$

of the nth degree.

330. **(a)** The sum of the roots of the equation

$$x^n - \frac{C(2n+1,3)}{C(2n+1,\ 1)}x^{n-1} + \frac{C(2n+1,5)}{C(2n+1,\ 1)}x^{n-2} - \ldots = 0$$

of the nth degree (see the solution of Problem 229 (b)) is equal to minus the coefficient in x^{n-1}, that is

$$\cot^2\frac{\pi}{2n+1}+\cot^2\frac{2\pi}{2n+1}+\cot^2\frac{3\pi}{2n+1}+\dots$$
$$\dots+\cot^2\frac{n\pi}{2n+1}=\frac{C(2n+1,3)}{C(2n+1,1)}=\frac{n(2n-1)}{3}$$

(b) Since $\csc^2\alpha=\cot^2\alpha+1$, the formula of Problem 330 (a) implies

$$\csc^2\frac{\pi}{2n+1}+\csc^2\frac{2\pi}{2n+1}+\csc^2\frac{3\pi}{2n+1}+\dots$$
$$\dots+\csc^2\frac{n\pi}{2n+1}=\frac{n(2n-1)}{3}+n=\frac{2n(n+1)}{3}$$

331. **(a)** *First solution.* The numbers $\sin^2\pi/(2n+1)$, $\sin^2 2\pi/(2n+1)$, ..., $\sin^2 n\pi/(2n+1)$ are the roots of the equation of the nth degree obtained in the solution of Problem 329 (a). In this equation the leading coefficient (the coefficient in x^n) is equal to $(-1)^n\,[C(2n+1,1)+C(2n+1,3)+\dots+C(2n+1,2n-1)+1]$. The sum in the square brackets equals half the total sum of the binomial coefficients $1+C(2n+1,1)+C(2n+1,2)+\dots+C(2n+1,2n)+1$; as is known, the latter sum is equal to $(1+1)^{2n+1}=2^{2n+1}$. Consequently, the coefficient in x^n of this equation is equal to $(-1)^n2^{2n}$. Further, the constant term of the equation is equal to $C(2n+1,1)=2n+1$. The product of the roots of an equation of the nth degree becomes equal to the constant term of the equation after the equation is brought to the form in which the leading coefficient is equal to unity times $(-1)^n$. Hence this product is equal to the constant term $2n+1$ multiplied by $(-1)^n$ and divided by the above leading coefficient, whence

$$(-1)^n\sin^2\frac{\pi}{2n+1}\sin^2\frac{2\pi}{2n+1}\dots\sin^2\frac{n\pi}{2n+1}=(-1)^n\frac{2n+1}{2^{2n}}$$

and, consequently,

$$\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\dots\sin\frac{n\pi}{2n+1}=\frac{\sqrt{2n+1}}{2^n}$$

It can similarly be proved that

$$\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\dots\sin\frac{(n-1)\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$$

Second solution. The roots of the equation $x^{2n}-1=0$ are

$$1,\quad -1,\quad \cos\frac{\pi}{n}+i\sin\frac{\pi}{n},\quad \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},$$
$$\cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n},\ \dots,\ \cos\frac{(2n-1)\pi}{n}+i\sin\frac{(2n-1)\pi}{n}$$

Therefore

$$x^{2n}-1=(x-1)(x+1)\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\times$$
$$\times\left(x-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right)\dots\left(x-\cos\frac{(n-1)\pi}{n}-i\sin\frac{(n-1)\pi}{n}\right)\times$$
$$\times\left(x-\cos\frac{(n+1)\pi}{n}-i\sin\frac{(n+1)\pi}{n}\right)\times\dots$$
$$\dots\times\left(x-\cos\frac{(2n-1)\pi}{n}-i\sin\frac{(2n-1)\pi}{n}\right)$$

Further, since $\cos(2n-k)\pi/n=\cos k\pi/n$ and $\sin(2n-k)\pi/n=-\sin k\pi/n$, it follows that

$$\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\left(x-\cos\frac{(2n-1)\pi}{n}-i\sin\frac{(2n-1)\pi}{n}\right)=x^2-2x\cos\frac{\pi}{n}+1$$

$$\left(x-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right)\left(x-\cos\frac{(2n-2)\pi}{n}-i\sin\frac{(2n-2)\pi}{n}\right)=x^2-2x\cos\frac{2\pi}{n}+1$$

. . . .

$$\left(x-\cos\frac{(n-1)\pi}{n}-i\sin\frac{(n-1)\pi}{n}\right)\times\left(x-\cos\frac{(n+1)\pi}{n}-i\sin\frac{(n+1)\pi}{n}\right)=x^2-2x\cos\frac{(n-1)\pi}{n}+1$$

Therefore the factorization of the polynomial $x^{2n}-1$ can be rewritten thus:

$$x^{2n}-1=(x^2-1)\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\times\dots\times\left(x^2-2x\cos\frac{(n-1)\pi}{n}+1\right)$$

It follows that

$$\frac{x^{2n}-1}{x^2-1}=x^{2n-2}+x^{2n-4}+\dots+x^2+1=$$
$$=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\times\dots\times\left(x^2-2x\cos\frac{(n-1)\pi}{n}+1\right)$$

Putting $x=1$ here and using the relation $2-2\cos\alpha=4\sin^2\alpha/2$ we obtain

$$n=4^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{2\pi}{2n}\dots\sin^2\frac{(n-1)\pi}{2n}$$

whence it follows that

$$\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$$

It can similarly be proved that

$$\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\ldots\sin\frac{n\pi}{2n+1}=\frac{\sqrt{2n+1}}{2^n}$$

(b) The required result can be obtained by complete analogy with the first and the second solutions of Problem 331 (a); however we shall not repeat the course of these solutions. Let us derive the required formulas directly from the formulas of Problem 331 (a).

Since

$$\sin\frac{\pi}{2n+1}=\sin\frac{2n\pi}{2n+1},\quad \sin\frac{3\pi}{2n+1}=\sin\frac{(2n-2)\pi}{2n+1},$$
$$\sin\frac{5\pi}{2n+1}=\sin\frac{(2n-4)\pi}{2n+1},\ \ldots$$

we have

$$\sin\frac{2\pi}{2n+1}\sin\frac{4\pi}{2n+1}\sin\frac{6\pi}{2n+1}\ldots\sin\frac{2n\pi}{2n+1}=$$
$$=\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\sin\frac{3\pi}{2n+1}\ldots\sin\frac{n\pi}{2n+1}=\frac{\sqrt{2n+1}}{2^n}$$

(see Problem 331 (a)). On performing the termwise division of the last formula by $\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\ldots\sin\frac{n\pi}{2n+1}=\frac{\sqrt{2n+1}}{2^n}$ and using the relations

$$\sin\frac{2\pi}{2n+1}=2\sin\frac{\pi}{2n+1}\cos\frac{\pi}{2n+1},\quad \sin\frac{4\pi}{2n+1}=$$
$$=2\sin\frac{2\pi}{2n+1}\cos\frac{2\pi}{2n+1},\ \ldots,\ \sin\frac{2n\pi}{2n+1}=2\sin\frac{n\pi}{2n+1}\cos\frac{n\pi}{2n+1}$$

we obtain

$$\cos\frac{\pi}{2n+1}\cos\frac{2\pi}{2n+1}\cos\frac{3\pi}{2n+1}\ldots\cos\frac{n\pi}{2n+1}=\frac{1}{2^n}$$

Similarly,

$$\left(\cos\frac{\pi}{2n}\cos\frac{2\pi}{2n}\ldots\cos\frac{(n-1)\pi}{2n}\right)\left(\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}\right)=$$
$$=\frac{1}{2^{n-1}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}$$

Since $\sin\pi/n = \sin(n-1)\pi/n$, $\sin 2\pi/n = \sin(n-2)\pi/n$, ..., $\sin\pi/2 = 1$, we have

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} =$$
$$= \left(\sin\frac{\pi}{2k+1}\sin\frac{2\pi}{2k+1}\ldots\sin\frac{k\pi}{2k+1}\right)^2 =$$
$$= \left(\frac{\sqrt{2k+1}}{2^k}\right)^2 = \frac{n}{2^{n-1}} \quad \text{for an odd } n = 2k+1$$

and

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} =$$
$$= \left(\sin\frac{\pi}{2k}\sin\frac{2\pi}{2k}\ldots\sin\frac{(k-1)\pi}{2k}\right)^2 =$$
$$= \left(\frac{\sqrt{k}}{2^{k-1}}\right)^2 = \frac{n}{2^{n-1}} \quad \text{for an even } n = 2k$$

(see Problem 331 (a)). It follows that

$$\cos\frac{\pi}{2n}\cos\frac{2\pi}{2n}\ldots\cos\frac{(n-1)\pi}{2n} = \frac{1}{2^{n-1}}\,\frac{\dfrac{n}{2^{n-1}}}{\dfrac{\sqrt{n}}{2^{n-1}}} = \frac{\sqrt{n}}{2^{n-1}}$$

Remark. On dividing the formulas of Problems 331 (a) and (b) by each other we obtain

$$\tan\frac{\pi}{2n+1}\tan\frac{2\pi}{2n+1}\ldots\tan\frac{n\pi}{2n+1} = \sqrt{2n+1}$$

and

$$\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\ldots\tan\frac{(n-1)\pi}{2n} = 1$$

It should be noted that the second of these relations is quite evident because $\tan k\pi/2n \tan(n-k)\pi/2n = \tan k\pi/2n \cot k\pi/2n = 1$ for $k = 1, 2, \ldots, n-1$ and $\tan\pi/4 = 1$. From this relation and from the second formula of Problem 331 (a) we can derive in a simple manner the formula $\cos\pi/2n\cos 2\pi/2n\ldots$ $\ldots\cos(n-1)\pi/2n = \dfrac{\sqrt{n}}{2^{n-1}}$.

These formulas can also be obtained by analogy with the first solution of Problem 331 (a).

332. Let us show that for any positive angle α smaller than $\pi/2$ we have

$$\sin\alpha < \alpha < \tan\alpha$$

To this end we consider Fig. 39 from which it is seen that

$$S_{\Delta AOB}=\frac{1}{2}\sin\alpha$$

$$S_{\text{sector } AOB}=\frac{1}{2}\alpha$$

$$S_{\Delta AOC}=\frac{1}{2}\tan\alpha$$

(in Fig. 39 it is meant that the radius of the circle is equal to unity and the angles are measured in radians). Since $S_{\Delta AOB}<S_{\text{sector } AOB}<S_{\Delta AOC}$ we conclude that $\sin\alpha<\alpha<\tan\alpha$.

The inequalities $\sin\alpha<\alpha<\tan\alpha$ imply $\cot\alpha<1/\alpha<\csc\alpha$. Therefore from the formulas of Problems 330 (a) and (b) it follows that

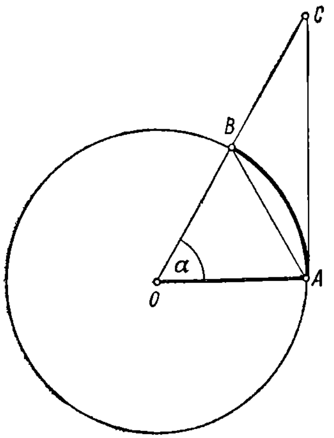


Fig. 39

$$\frac{n(2n-1)}{3}=\cot^2\frac{\pi}{2n+1}+\cot^2\frac{2\pi}{2n+1}+\cot^2\frac{3\pi}{2n+1}+\ldots+\cot^2\frac{n\pi}{2n+1}<$$

$$<\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\left(\frac{2n+1}{3\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2<$$

$$<\csc^2\frac{\pi}{2n+1}+\csc^2\frac{2\pi}{2n+1}+\csc^2\frac{3\pi}{2n+1}+\ldots+\csc^2\frac{n\pi}{2n+1}=\frac{2n(n+1)}{3}$$

On dividing all the members of the last inequalities by $(2n+1)^2/\pi^2$ we obtain

$$\frac{2n}{2n+1}\cdot\frac{2n-1}{2n+1}\cdot\frac{\pi^2}{6}=\left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)\cdot\frac{\pi^2}{6}<1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

$$\ldots+\frac{1}{n^2}<\frac{2n}{2n+1}\cdot\frac{2n+2}{2n+1}\cdot\frac{\pi^2}{6}=\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right)\cdot\frac{\pi^2}{6}$$

which is what we had to prove.

333. (a) Let us suppose that M is a point on an arc A_1A_n of the circle depicted in Fig. 40. We shall denote the arc MA_1 by α;

then the arcs $MA_2, MA_3, \ldots, MA_n$ are equal to

$$\alpha+\frac{2\pi}{n}, \quad \alpha+\frac{4\pi}{n}, \ldots, \alpha+\frac{2(n-1)\pi}{n}$$

respectively. The length of the chord AB of a circle of radius R is equal to $2R \sin \overset{\smile}{AB}/2$ (this can readily be seen from the isosceles

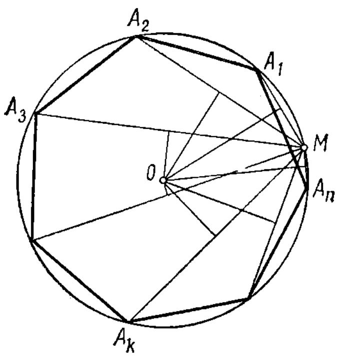


Fig. 40

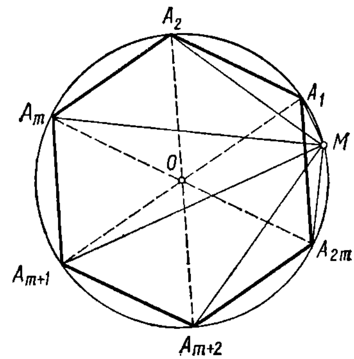


Fig. 41

triangle AOB where O is the centre of the circle). It follows that the sum we are interested in is equal to

$$4R^2\left[\sin^2\frac{\alpha}{2}+\sin^2\left(\frac{\alpha}{2}+\frac{\pi}{n}\right)+\sin^2\left(\frac{\alpha}{2}+\frac{2\pi}{n}\right)+\ldots\right.$$
$$\left.\ldots+\sin^2\left(\frac{\alpha}{2}+\frac{(n-1)\pi}{n}\right)\right]$$

Now let us compute the expression in the square brackets. Using the well-known formula $\sin^2 x=(1-\cos 2x)/2$ we find that this expression is equal to

$$S=\frac{n}{2}-\left[\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cos\left(\alpha+\frac{4\pi}{n}\right)+\ldots\right.$$
$$\left.\ldots+\cos\left(\alpha+\frac{2(n-1)\pi}{n}\right)\right]$$

By the formula of Problem 324, we have

$$\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\ldots+\cos\left(\alpha+\frac{2(n-1)\pi}{n}\right)=$$
$$=\frac{\sin\pi\cos\left(\alpha+\frac{(n-1)\pi}{n}\right)}{\sin\frac{\pi}{n}}=0$$

and, consequently, $S=n/2$, whence follows the assertion stated in the condition of the problem.

Remark. The assertion of the problem is quite evident for an even $n = 2m$ (see Fig. 41) because, by Pythagoras' theorem, we have

$$MA_1^2 + MA_{m+1}^2 = MA_2^2 + MA_{m+2}^2 = \ldots = MA_m^2 + MA_{2m}^2 = 4R^2$$

(b) Let $A_1B_1, A_2B_2, \ldots, A_nB_n$ (see Fig. 42(*a*)) be the perpendiculars dropped from the points $A_1, A_2, \ldots, A_n$ on the straight line OM. Then, according to the well-known theorem of plane geometry, we have

$$MA_k^2 = MO^2 + OA_k^2 - 2MO \cdot OB_k = l^2 + R^2 - 2l \cdot OB_k$$

where the line segment OB_k ($k = 1, 2, \ldots, n$) is taken with the sign "+" or "—" depending on whether the point B_k lies on the ray OM or on its extension to the left of the point O. Consequently,

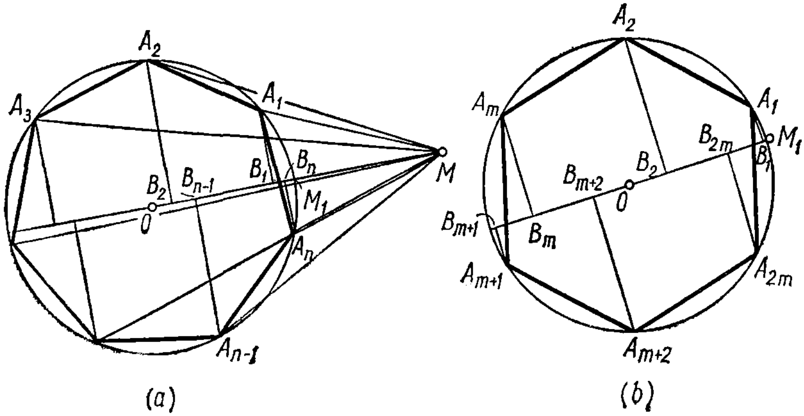


Fig. 42

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = n(l^2 + R^2) - 2l(OB_1 + OB_2 + \ldots + OB_n)$$

Let $\angle MOA_1 = \alpha$ then

$$OB_1 = OA_1 \cos \angle A_1OM = R\cos\alpha, \qquad OB_2 = R\cos\left(\alpha + \frac{2\pi}{n}\right),$$

$$OB_3 = R\cos\left(\alpha + \frac{4\pi}{n}\right), \quad \ldots, \quad OB_n = R\cos\left(\alpha + \frac{2(n-1)\pi}{n}\right)$$

Since in the solution of Problem 333 (a) it was shown that

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \ldots + \cos\left(\alpha + \frac{2(n-1)\pi}{n}\right) = 0$$

we have $OB_1 + OB_2 + \ldots + OB_n = 0$, whence follows the assertion of the problem.

Remark. For an even $n = 2m$ (see Fig. 42(b)) the assertion of the problem can be proved purely geometrically because in this case

$$OB_1 + OB_{m+1} = OB_2 + OB_{m+2} = OB_m + OB_{2m} = 0$$

(c) Let us consider Fig. 43 where M_1 is the projection of the point M on the plane in which the n-gon $A_1A_2 \ldots A_{n-1}A_n$ lies. Then we have $MA_k^2 = M_1A_k^2 + MM_1^2$ ($k = 1, 2, \ldots, n$), and consequently

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = \\ = M_1A_1^2 + M_1A_2^2 + \ldots \\ \ldots + M_1A_n^2 + n \cdot MM_1^2$$

By Problem 333 (b), we have $M_1A_1^2 + M_1A_2^2 + \ldots + M_1A_n^2 = n(R^2 + OM_1^2)$, and $l^2 = OM^2 = OM_1^2 + M_1M^2$, whence follows the assertion of the problem.

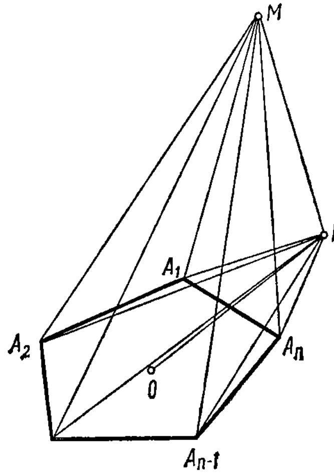


Fig. 43

334. (a) The assertion stated in this problem is a direct consequence of the theorem proved in the solution of Problem 333 (a) because for an even n the vertices of the n-gon having even indices and those with odd indices are themselves the vertices of the corresponding regular $n/2$-gons inscribed in the circle.

(b) Let $n = 2m + 1$. From the solution of Problem 333 (a) it is readily seen that it suffices to show that the sums

$$S_1 = \sin\frac{\alpha}{2} + \sin\left(\frac{\alpha}{2} + \frac{2\pi}{2m+1}\right) + \sin\left(\frac{\alpha}{2} + \frac{4\pi}{2m+1}\right) + \ldots \\ \ldots + \sin\left(\frac{\alpha}{2} + \frac{2m\pi}{2m+1}\right)$$

and

$$S_2 = \sin\left(\frac{\alpha}{2} + \frac{\pi}{2m+1}\right) + \sin\left(\frac{\alpha}{2} + \frac{3\pi}{2m+1}\right) + \ldots \\ \ldots + \sin\left(\frac{\alpha}{2} + \frac{(2m-1)\pi}{2m+1}\right)$$

are equal. To this end we note that, according to Problem 324, we have

$$S_1 = \frac{\sin\frac{(m+1)\pi}{2m+1}\sin\left(\frac{\alpha}{2} + \frac{m\pi}{2m+1}\right)}{\sin\frac{\pi}{2m+1}}$$

and

$$S_2 = \frac{\sin \frac{m\pi}{2m+1} \sin\left(\frac{\alpha}{2} + \frac{\pi}{2m+1} + \frac{(m-1)\pi}{2m+1}\right)}{\sin \frac{\pi}{2m+1}} = S_1$$

Thus, the theorem has been proved.

335. (a) By virtue of Problem 333 (a), the sum of the squares of the distances from a point on the circle circumscribed about a regular n-gon to all its vertices is equal to $2nR^2$. Assuming that M coincides with A_1 we conclude that the sum of the squares of all the sides and diagonals of the n-gon issued from one vertex is equal to $2nR^2$. The multiplication of this sum by n results in twice the sum of the squares of all the sides and diagonals of the n-gon (since every side and every diagonal has two end points, it is involved twice in that sum). It follows that the sought-for sum is equal to $(n/2) \cdot 2nR^2 = n^2R^2$.

(b) The sum of all the sides and diagonals issued from one vertex A_1 of a regular n-gon is equal to

$$2R\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{(n-1)\pi}{n}\right] = 2R\frac{\sin\frac{\pi}{2}\sin\frac{(n-1)\pi}{2n}}{\sin\frac{\pi}{2n}} = 2R\cot\frac{\pi}{2n}$$

(cf. Problem 334 (b)). On multiplying this sum by n and taking half that product we obtain the required result $Rn\cot\pi/2n$.

(c) The product of all the sides and diagonals issuing from one vertex of a regular n-gon inscribed in a circle of radius R is obviously equal to

$$2^{n-1}R^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = 2^{n-1}R^{n-1}\frac{n}{2^{n-1}}$$

(cf. Problem 331 (a)). On raising this product to the nth power and extracting the square root of the result we obtain the required expression.

336. Let us compute the sum of the 50th powers of all the sides and diagonals issued from one vertex A_1 of a regular 100-gon inscribed in a circle of radius R. The problem reduces to the determination of the sum

$$\sum = \left(2R\sin\frac{\pi}{100}\right)^{50} + \left(2R\sin\frac{2\pi}{100}\right)^{50} + \ldots + \left(2R\sin\frac{99\pi}{100}\right)^{50}$$

(cf. the solution of Problem 333 (a)). Thus, we have to add together the 50th powers of the sines of the angles $\pi/100$, $2\pi/100$, ..., $99\pi/100$.

We have

$$\sin^{50}\alpha = \left(\frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{2i}\right)^{50} =$$

$$= \frac{\left(x - \frac{1}{x}\right)^{50}}{-2^{50}} = -\frac{1}{2^{50}}\left(x - \frac{1}{x}\right)^{50}$$

where x denotes $\cos\alpha + i\sin\alpha$ (if $x = \cos\alpha + i\sin\alpha$ then $1/x = \cos\alpha - i\sin\alpha$). Consequently,

$$\sin^{50}\alpha = \frac{-1}{2^{50}}\left(x^{50} - C(50,\ 1)\,x^{49}\frac{1}{x} + C(50,\ 2)\,x^{48}\frac{1}{x^2} + \ldots\right.$$

$$\ldots + C(50,\ 24)\,x^{26}\frac{1}{x^{24}} - C(50,\ 25)\,x^{25}\frac{1}{x^{25}} + C(50,\ 26)\,x^{24}\frac{1}{x^{26}} + \ldots$$

$$\left.\ldots + C(50,\ 48)\,x^2\frac{1}{x^{48}} - C(50,\ 49)\,x\frac{1}{x^{49}} + \frac{1}{x^{50}}\right) =$$

$$= -\frac{1}{2^{50}}\left[\left(x^{50} + \frac{1}{x^{50}}\right) - C(50,\ 1)\left(x^{48} + \frac{1}{x^{48}}\right) + \right.$$

$$\left. + C(50, 2)\left(x^{46} + \frac{1}{x^{46}}\right) - \ldots + C(50, 24)\left(x^2 + \frac{1}{x^2}\right) - C(50, 25)\right] =$$

$$= -\frac{1}{2^{50}}(2\cos 50\alpha - 2C(50,\ 1)\cos 48\alpha + 2C(50,\ 2)\cos 46\alpha - \ldots$$

$$\ldots + 2C(50,\ 24)\cos 2\alpha + C(50,\ 25))$$

(here we have used the relation $x^k + \frac{1}{x^k} = (\cos k\alpha + i\sin k\alpha) + (\cos k\alpha - i\sin k\alpha) = 2\cos k\alpha$).

Thus, the sum Σ can be rewritten in the following way:

$$\sum = -R^{50}\left[2\left(\cos 50\frac{\pi}{100} + \cos 50\frac{2\pi}{100} + \ldots + \cos 50\frac{99\pi}{100}\right) - \right.$$

$$- 2C(50,\ 1)\left(\cos 48\frac{\pi}{100} + \cos 48\frac{2\pi}{100} + \ldots + \cos 48\frac{99\pi}{100}\right) +$$

$$+ 2C(50,\ 2)\left(\cos 46\frac{\pi}{100} + \cos 46\frac{2\pi}{100} + \ldots + \cos 46\frac{99\pi}{100}\right) -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left. + 2C(50,\ 24)\left(\cos 2\frac{\pi}{100} + \cos 2\frac{2\pi}{100} + \ldots + \cos 2\frac{99\pi}{100}\right) - 99C(50, 25)\right] =$$

$$= -R^{50}[2s_1 - 2C(50,\ 1)\,s_2 + 2C(50,\ 2)\,s_3 - \ldots$$

$$\ldots + 2C(50,\ 24)\,s_{25} - 99C(50,\ 25)]$$

where the letters $s_1, s_2, \ldots, s_{25}$ designate the sums in the parentheses. The formulas of Problem 324 readily imply that $s_1 = s_2 = \ldots$

$\ldots = s_{25} = -1$. Consequently,

$$\textstyle\sum' = R^{50}(2-2C(50,1)+2C(50,2)-\ldots+2C(50,24)+99C(50,25) =$$
$$= R^{50}(1-C(50,1)+C(50,2)-C(50,3)+\ldots$$
$$\ldots+C(50,24)-C(50,25)+C(50,26)-\ldots$$
$$\ldots+C(50,48)-C(50,49)+1+100C(50,25)) =$$
$$= R^{50}[(1-1)^{500}+100C(50,25)] = 100C(50,25)R^{50}$$

It readily follows that the sum of the 50th powers of all the sides and all the diagonals of the 100-gon is equal to

$$\frac{100\sum}{2} = 5000C(50,25)R^{50} = \frac{5000\cdot 50!}{(25!)^2}R^{50}$$

337. Since $|z|=|\bar z|=|-z|=|-\bar z|$ and $\left|z+\frac{1}{z}\right|=\left|\bar z+\frac{1}{\bar z}\right|=$ $=\left|-z-\frac{1}{z}\right|=\left|-\bar z-\frac{1}{\bar z}\right|$, it is sufficient to consider only one of the numbers, z, $\bar z$, $-z$ and $-\bar z$, namely the one *lying in the first quadrant.* When $|z|$ assumes its maximum possible value the expression $|1/z|=1/|z|$ assumes the minimum value. Therefore it suffices to find those z whose modulus assumes the *greatest* possible value under the assumption that $|z|\geqslant|1/z|$. Let the argument of the number z be φ $(0\leqslant\varphi\leqslant\pi/2$; (see Fig. 44). Since $|z+1/z|=a$ we can write the relation

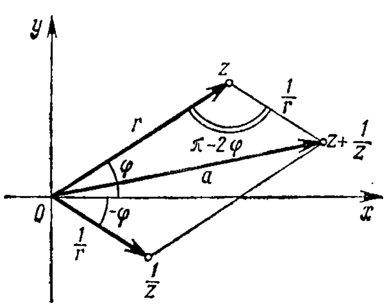


Fig. 44

$$a^2 = r^2+\frac{1}{r^2}-2\cos 2\varphi =$$
$$= r^2+\frac{1}{r^2}-2+4\cos^2\varphi =$$
$$= \left(r-\frac{1}{r}\right)^2+4\cos^2\varphi$$

where r denotes $|z|$. By the hypothesis, we have $r\geqslant 1/r$, and therefore when r increases the difference $r-1/r$ decreases and vice versa. Further, we have $(r-1/r)^2=a^2-4\cos^2\varphi\leqslant a^2$; for $\varphi=\pi/2$ we obtain $(r-1/r)^2=a^2$, and in this case $r-1/r=a$ and $r=(a+\sqrt{a^2+4})/2$.

It follows that the greatest value $|z|=(a+\sqrt{a^2+4})/2$ is attained for $z=i(a+\sqrt{a^2+4})/2$ and the smallest value $|z|=$ $=(\sqrt{a^2+4}-a)/2$ for $z=-i(\sqrt{a^2+4}-a)/2$.

338. It is clear that the complex numbers $1 = 1 + i \cdot 0 = \cos 0° + i \sin 0°$, $(-1 + i\sqrt{3})/2 = \cos 120° + i \sin 120°$ and $(-1 - i\sqrt{3})/2 = \cos(-120°) + i \sin(-120°)$ are such that the arguments of every two of them differ by exactly 120° and the sum of the three numbers is equal to zero. This means that the angle of 120° mentioned in the condition of the problem cannot be replaced by a *greater* one.

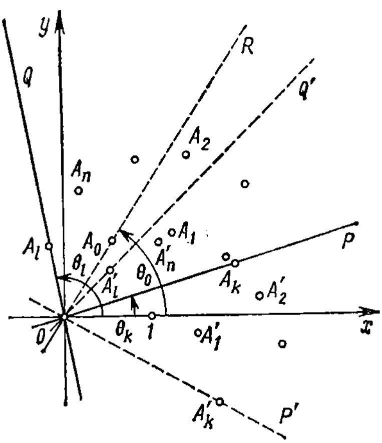


Fig. 45

Now let $|\theta_i - \theta_j| < 120°$ where $i, j = 1, 2, \ldots, n$ and $i \neq j$ (here $\theta_1, \theta_2, \ldots, \theta_n$ are the arguments of the given complex numbers; it is evident that if one of these numbers is equal to zero and therefore possesses no definite argument it can simply be discarded). We shall prove that in this case the equality $z_1 + z_2 + \ldots + z_n = 0$ *cannot be fulfilled*. Indeed, from the hypothesis we have stated it follows that the points A_1, $A_2, \ldots, A_n$ in the complex plane with polar coordinates (r_1, θ_1), $(r_2, \theta_2), \ldots, (r_n, \theta_n)$ representing the complex numbers $z_1, z_2, \ldots, z_n$ lie within an angle $\angle POQ$ bounded by two rays $\theta = \theta_k$ and $\theta = \theta_l$, this angle being smaller than 120°. Here A_k and A_l are the "extreme" points which correspond to the numbers z_k and z_l with the "extreme" values of the argument (see Fig. 45). Let the ray OR corresponding to a value $\theta = \theta_0$ of the polar angle be the *bisector* of the angle $\angle POQ$ and let $z_0 = \cos\theta_0 + i \sin\theta_0$ be a complex number of unit modulus represented by a point A_0 lying on that ray. Since the numbers $z'_1 = (z_1/z_0)[r_1 \cos(\theta - \theta_0) + i \sin(\theta - \theta_0)]$, $z'_2 = z_2/z_0, \ldots, z'_n = z_n/z_0$ have the same absolute values $r_1, r_2, \ldots, r_n$ as the numbers $z_1, z_2, \ldots, z_n$ and have the arguments $\theta_1 - \theta_0, \theta_2 - \theta_0, \ldots, \theta_n - \theta_0$ instead of the arguments $\theta_1, \theta_2, \ldots, \theta_n$ of the numbers $z_1, z_2, \ldots, z_n$, the points $A'_1, A'_2, \ldots, A'_n$ in the complex plane representing the numbers $z'_1, z'_2, \ldots, z'_n$ are obtained from the points $A_1, A_2, \ldots, A_n$ by the rotation of the latter about the point O through an angle of θ_0 in the clockwise direction. It follows that all these points lie within the angle $\angle P'OQ'$ smaller than 120° obtained from $\angle POQ$ under that rotation, the bisector OR' of $\angle P'OQ'$ coinciding with the real axis Ox. Further, it follows that the real parts $a'_1, a'_2, \ldots, a'_n$ of the numbers $z'_1 = a'_1 + ib'_1$, $z'_2 =$

$= a_2' + ib_2', \ldots, z_n' = a_n' + ib_n'$ are all *positive*. Therefore the sum

$$z_1' + z_2' + \ldots + z_n' = (a_1' + ib_1') + (a_2' + ib_2') + \ldots + (a_n' + ib_n') = \\ = (a_1' + a_2' + \ldots + a_n') + i(b_1' + b_2' + \ldots + b_n')$$

cannot be equal to zero (because $a_1' + a_2' + \ldots + a_n' > 0$). Now, from the relation

$$z_1' + z_2' + \ldots + z_n' = \frac{z_1}{z_0} + \frac{z_2}{z_0} + \ldots + \frac{z_n}{z_0} = \\ = (z_1 + z_2 + \ldots + z_n)\frac{1}{z_0} \neq 0$$

it follows that the sum $z_1 + z_2 + \ldots + z_n$ is also different from zero.

339. Let us suppose that the point A in the complex plane representing the complex number z does not lie within the convex polygon $C_1, C_2, \ldots, C_n$ whose vertices correspond to the numbers $c_1, c_2, \ldots, c_n$ (see Fig. 46). In this case all the rays $AC_1, AC_2, \ldots, AC_n$ "go in one direction" in the sense that they all lie on one side of a straight line l passing through A. According to the subtraction rule for complex numbers, the numbers $z_1 = z - c_1$, $z_2 = z - c_2, \ldots, z_n = z - c_n$ are represented by some points $A_1, A_2, \ldots, A_n$ in the complex plane such that the vectors $\overline{OA_1}, \overline{OA_2}, \ldots, \overline{OA_n}$ are equal to the vectors $\overline{C_1A}, \overline{C_2A}, \ldots \overline{C_nA}$ respectively. Therefore all the rays $OA_1, OA_2, \ldots, OA_n$ lie on one side of the straight line l' passing through O and parallel to l. Further, if $w' = 1/w$ and $w = r\ (\cos\theta + i\sin\theta)$ then $w' = (1/r)(\cos(-\theta) + i\sin(-\theta))$, that is the numbers w and w' are represented by points B and B' in the complex plane such that the rays OB and OB' are symmetric about the axis Ox of reals. It follows that the numbers $z_1' = 1/z_1 = 1/(z - c_1)$, $z_2' = 1/z_2, \ldots, z_n' = 1/z_n$ are represented in the complex plane by points A_1', $A_2', \ldots, A_n'$ such that the rays $OA_1', OA_2', \ldots, OA_n'$ are symmetric to the rays $OA_1, OA_2, \ldots, OA_n$ with respect to the axis Ox, whence it follows that all these rays lie on one side of a straight line l'' symmetric to l' about the axis Ox (see again Fig. 46).

The further course of the proof is rather close to the solution of Problem 338. Let $z_0 = \cos\theta_0 + i\sin\theta_0$ be a complex number of unit modulus represented by a point lying on the straight line l''; then the numbers $z_1'' = z_1'/z_0$, $z_2'' = z_2'/z_0, \ldots, z_n'' = z_n'/z_0$ are represented by the points $A_1'', A_2'', \ldots, A_n''$ obtained from the points $A_1', A_2', \ldots, A_n'$ by a rotation about O through an angle of θ_0. Therefore all these points $A_1'', A_2'', \ldots, A_n''$ lie on one side of the axis Ox which is obtained with the aid of the same rotation from the straight line l''. It follows that the imaginary parts $b_1'', b_2'', \ldots$

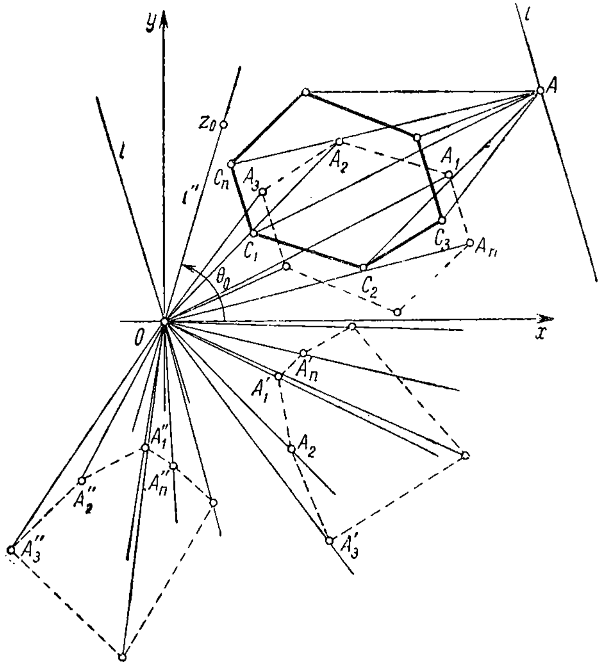


Fig. 46

$\ldots, b_n''$ of the numbers $z_1'' = a_1'' + ib_1''$, $z_2'' = a_2'' + ib_2''$, $\ldots$, $z_n'' = a_n'' + ib_n''$ are all of one sign, and therefore

$$z_1'' + z_2'' + \ldots + z_n'' = (a_1'' + a_2'' + \ldots + a_n'') + i(b_1'' + b_2'' + \ldots + b_n'') \neq 0$$

because $b_1'' + b_2'' + \ldots + b_n'' \neq 0$. Further, from the relation

$$z_1'' + z_2'' + \ldots + z_n'' = \frac{z_1'}{z_0} + \frac{z_2'}{z_0} + \ldots + \frac{z_n'}{z_0} = (z_1' + z_2' + \ldots + z_n') \cdot \frac{1}{z_0} \neq 0$$

it also follows that

$$z_1' + z_2' + \ldots + z_n' = \frac{1}{z - c_1} + \frac{1}{z - c_2} + \ldots + \frac{1}{z - c_n} \neq 0$$

340. *First solution.* Suppose that a is not divisible by p. Then the numbers $a, 2a, 3a, \ldots, (p-1)a$ are not divisible by p either, and their division by p leaves *different* remainders. For, if the division of ka and la (where $p-1 \geqslant k > l$) by p left equal remainders then the difference $ka - la = (k - l)a$ would be divisible by p, which is impossible since p is a prime number, the number a is not divisible by p and the difference $k - l$ is less than p. Since the set of the possible remainders resulting from the division by p are exhausted by the $p - 1$ numbers $1, 2, 3, \ldots, p - 1$, there must be

$$a = q_1 p + a_1, \quad 2a = q_2 p + a_2, \quad 3a = q_3 p + a_3, \quad \ldots, \quad (p-1)a = \\ = q_{p-1} p + a_{p-1}$$

where $a_1, a_2, \ldots, a_{p-1}$ are the numbers $1, 2, \ldots, p - 1$ taken in some order. On multiplying all these equalities we obtain

$$[1 \cdot 2 \cdot \ \ldots \ \cdot (p-1)]\, a^{p-1} = Np + a_1 a_2 \ldots a_{p-1}$$

that is

$$[1 \cdot 2 \cdot \ \ldots \ \cdot (p-1)]\, (a^{p-1} - 1) = Np$$

It follows that $a^{p-1} - 1$ is divisible by p and, consequently, $a^p - a$ is also divisible by p. In case a is divisible by p the assertion of Fermat's theorem is evident.

Second solution. The theorem is evidently true for $a = 1$ because in this case the difference $a^p - a = 1 - 1 = 0$ is divisible by any number. Now we shall prove the theorem *by induction*: let us assume that it is already known that $a^p - a$ is divisible by p and prove that under this assumption $(a + 1)^p - (a + 1)$ is also divisible by p.

By Newton's binomial formula, we have

$$(a+1)^p - (a+1) = \\ = a^p + pa^{p-1} + C(p, 2)\, a^{p-2} + C(p, 3)\, a^{p-3} + \ldots + pa + 1 - a - 1 = \\ = (a^p - a) + pa^{p-1} + C(p, 2)\, a^{p-2} + \ \ldots \ + C(p, p-2)\, a^2 + pa$$

Further, every binomial coefficient

$$C(p, k) = \frac{p(p-1)(p-2)\ \ldots\ (p-k+1)}{1 \cdot 2 \cdot 3 \ \ldots \ k}$$

is divisible by the prime number p since the numerator of this expression contains the factor p while the denominator does not. Finally, by the hypothesis, the number $a^p - a$ is divisible by p, and therefore $(a + 1)^p - (a + 1)$ is also divisible by p.

Remark. We shall present one more variant of the same proof. Since all the binomial coefficients $C(p, k)$ are divisible by p, the difference

$$(A+B)^p - A^p - B^p = \\ = pA^{p-1}B + C(p, 2)\, A^{p-2}B^2 + \ \ldots \ + C(p, p-2)\, A^2 B^{p-2} + pAB^{p-2}$$

where A and B are arbitrary integers is always divisible by p. On applying this result we consecutively find that

$$(A+B+C)^p - A^p - B^p - C^p =$$
$$= \{[(A+B)+C]^p - (A+B)^p - C^p\} + (A+B)^p - A^p - B^p$$

is always divisible by p,

$$(A+B+C+D)^p - A^p - B^p - C^p - D^p = \{[(A+B+C)+D]^p -$$
$$- (A+B+C)^p - D^p\} + (A+B+C)^p - A^p - B^p - C^p$$

is always divisible by p, and, generally,

$$(A+B+C+\ \ldots\ +K)^p - A^p - B^p - C^p - \ \ldots\ - K^p$$

is always divisible by p where all capital letters denote arbitrary integers.

Now, putting $A = B = C = \ldots = K = 1$ in the last relation and taking the total number of the integers equal to a we arrive at Fermat's theorem: $a^p - a$ is divisible by p.

341. The proof of Euler's theorem is completely analogous to the first proof of Fermat's theorem. Let us denote the r numbers smaller than N and relatively prime to N as $k_1, k_2, k_3, \ldots, k_r$. We shall consider the r numbers $k_1a, k_2a, \ldots, k_ra$. They all are relatively prime to N (because, by the condition of the problem, a and N are relatively prime), and their division by N leaves different remainders (the latter property is proved by complete analogy with the solution of Problem 340). It follows that

$$k_1a = q_1N + a_1, \quad k_2a = q_2N + a_2, \quad \ldots, \quad k_ra = q_rN + a_r$$

where $a_1, a_2, \ldots, a_r$ are the same numbers $k_1, k_2, \ldots, k_r$ possibly arranged in some other order. On multiplying all these equalities we obtain

$$k_1k_2 \ldots k_ra^r = MN + a_1a_2 \ldots a_r, \quad \text{that is} \quad k_1k_2 \ldots k_r(a^r - 1) = MN$$

where M is an integer, whence it follows that the number $a^r - 1$ is divisible by N.

342. We shall elaborate the proof *by induction*. It is obvious that the proposition stated in the condition of the problem is true for $n = 1$ because the numbers $2^1 - 1 = 1$, $2^2 - 1 = 3$ and $2^3 - 1 = 7$ are not divisible by 5. Let us also prove the proposition for $n = 2$. Let 2^k be the smallest power of the number 2 whose division by $5^2 = 25$ leaves a remaainder of 1 (that is $2^k - 1$ is exactly divisible by 25). Let us suppose that $k < 5^2 - 5 = 25 - 5 = 20$. If 20 is not divisible by k (in this case $20 = qk + r$ where $r < k$) then

$$2^{20} - 1 = 2^{qk+r} - 1 = 2^r(2^{qk} - 1) + (2^r - 1)$$

By Euler's theorem, $2^{20} - 1$ is divisible by 25; the number $2^{qk} - 1 = (2^k)^q - 1^q$ is divisible by $2^k - 1$ and therefore, by virtue of the hypothesis, it is divisible by 25 as well. Consequently, $2^r - 1$ is also divisible by 25, which contradicts the assumption that k is the *smallest* exponent for which $2^k - 1$ is divisible by 25. Thus, the number k must be a divisor of 20, that is k can only be equal to 2, 4, 5 or 10. Further, the numbers $2^2 - 1 = 3$, $2^5 - 1 = 31$ and $2^{10} - 1 = 1023$ are not divisible by 5 and therefore are not divisible by 25 either while the number $2^4 - 1 = 15$ is divisible by 5 but is not divisible by 25. Consequently, for $n = 2$ the proposition stated in the problem is also true.

Now let us suppose that the proposition holds for some n and does not hold for $n+1$; in other words, we suppose that the smallest exponent k such that $2^k - 1$ is divisible by 5^{n+1} is less than $5^{n+1} - 5^n = 4\cdot 5^n$. In just the same way as above (for $n = 2$) it is proved that the number k must be a divisor of the number $4\cdot 5^n$. At the same time, it is similarly proved that the number $5^n - 5^{n-1} = 4\cdot 5^{n-1}$ must be a divisor of the number k. Indeed, if we had $k = q\cdot 4\cdot 5^{n-1} + r$ where $r < 4\cdot 5^{n-1}$ then the number $5^r - 1$ would be divisible by 5^n, which contradicts the assumption that the proposition of the problem is true for the number n. Thus, the exponent k can assume a single possible value, namely $k = 4\cdot 5^{n-1}$.

By virtue of Euler's theorem, the number $2^{5^{n-1}-5^{n-2}} - 1 = 2^{4\cdot 5^{n-2}} - 1$ is divisible by 5^{n-1}; at the same time it is not divisible by 5^n (the latter property holds because, if otherwise, the proposition of the problem would not be true for the number n). Therefore we have $2^{4\cdot 5^{n-2}} = q\cdot 5^{n-1} + 1$ where q is not divisible by 5.

Now, using the formula

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

we obtain

$$2^{4\cdot 5^{n-1}} - 1 = \left(2^{4\cdot 5^{n-2}}\right)^5 - 1 = \left(q\cdot 5^{n-1} + 1\right)^5 - 1 =$$
$$= 5^{n+1}\left(q^5\cdot 5^{4n-6} + q^4\cdot 5^{3n-4} + 2q^3\cdot 5^{2n-3} + 2q^2\cdot 5^{n-2}\right) + q\cdot 5^n$$

whence it is seen that $2^{4\cdot 5^{n-2}} - 1$ is not divisible by 5^{n+1}. Thus, from the assumption that the proposition holds for some n it follows that it also holds for $n+1$.

343. According to Euler's theorem (see Problem 341), the number $2^{5^{10}-5^9} - 1 = 2^{4\cdot 5^9} - 1 = 2^{7\,812\,500} - 1$ is divisible by 5^{10}. Consequently, for $n \geqslant 10$ the difference $2^{7\,812\,500+n} - 2^n = 2^n(2^{7\,812\,500} - 1)$ is divisible by 10^{10}; therefore the last 10 digits of the numbers $2^{7\,812\,500+n}$ and 2^n coincide. This means that in the number sequence $2^1, 2^2, 2^3, \ldots, 2^n, \ldots$ the last 10 digits repeat with a period of 7 812 500 members of the sequence, the periodicity starting with the tenth member 2^{10} of this sequence.

The fact that the length of the period *is not less* than 7 812 500 follows from the result of Problem 342.

Remark. It can similarly be proved that the last n digits of the members of the sequence in question repeat with a period of $4\cdot 5^{n-1}$ members starting with the nth member (for instance, the last two digits repeat with a period of 20 members beginning with the second one).

344. We can even prove a more general proposition: *for any whole number N there always exists a power of the number* 2 *whose N last digits are all unities and twos.*

Since $2^5 = 32$ and $2^9 = 512$, the proposition is true for $N = 1$ and $N = 2$. In the further course of the proof we use the *method of mathematical induction*. Let us assume that the last N digits of the number 2^N are all unities and twos and prove that under this assumption there must exist a power of the number 2 whose last $N+1$ digits are unities and twos. By the hypothesis, we have $2^n = 10^N\cdot a + b$ where b is an N-digit number whose decimal representation contains only the two digits 1 and 2. Let us denote by r the number $5^N - 5^{N-1} = 4\cdot 5^{N-1}$; then, by Euler's theorem (see Problem 341), the difference $2^r - 1$ is divisible by 5^N. It follows that if an integer k is divisible by 2^{N+1} then the difference $2^rk - k = k(2^r - 1)$ must be divisible by $2\cdot 10^N$, that is the last N digits of the numbers 2^rk and k coincide and the $(N+1)$th (counting from right to left) digits of these numbers are simultaneously even or odd.

Now let us consider the following five powers of the number 2:

$$2^n, \quad 2^{n+r} = 2^r\cdot 2^n, \quad 2^{n+2r} = 2^r\cdot 2^{n+r}, \quad 2^{n+3r} = 2^r\cdot 2^{n+2r},$$
$$2^{n+4r} = 2^r\cdot 2^{n+3r}$$

According to what has been proved, the last N digits of all these numbers are the same, that is they all end with the same combination (number) b consisting of twos and unities, the number b coinciding with that in the representation $2^n = 10^Na + b$, while the $(N+1)$th (counting from right to left) digits of all these numbers are simultaneously even or odd. Next we shall prove that among these five powers of two there are not two numbers whose $(N+1)$th (counting from right to left) digits coincide. Indeed, the difference of any two of these powers can be represented in the form $2^{n+m_1r}(2^{m_2r} - 1)$ where $m_1 = 0$, 1, 2 or 3 and $m_2 = 1$, 2, 3, or 4. If this difference were divisible by 10^{N+1}, the number $2^{m_2r} - 1$ would be divisible by 5^{N+1}; however, since

$$m_2r = m_2\cdot(5^N - 5^{N-1}) < 5\cdot(5^N - 5^{N-1}) = 5^{N+1} - 5^N$$

this divisibility contradicts the result of Problem 342.

Thus, the $(N+1)$th (counting from right to left) digits of the five powers of two under consideration are either 1, 3, 5, 7 and 9 (these digits follow in some unknown order) or 0, 2, 4, 6 and 8. In both cases the $(N+1)$th (counting from right to left) digit of at least one of the powers is 1 or 2. Consequently, in all the cases there exists a power of the number 2 whose last $N+1$ digits are all unities and twos; by the principle of mathematical induction, it follows that the proposition we had to prove is true.

345. It is evident that a pair of the form n, n^2 where n is a natural number is "good" for any $n > 1$. Further, a pair of numbers $n-1$, $n^2-1=(n-1)(n+1)$ is sure to be "good" when the number $n+1$ is an *integral power of two*, that is $n+1=2^k$ where $k \geqslant 1$ is an integer. Indeed, in this case the only distinction between the numbers

$$n-1 \quad \text{and} \quad n^2-1=2^k(n-1)$$

is that the second of them involves a power of the prime factor two whose exponent exceeds by k that of the power of two contained in the first number (the number $n-1$ must contain the prime factor 2 because since $n+1=2^k$, that is $n=2^k-1$, the number $n-1=2^k-2$ is also even). It follows that there are an infinite number of "very good" pairs: for instance, such are all the pairs of numbers of the form

$$n-1=2^k-2, \quad n^2-1=(n-1)(n+1)=2^k(2^k-2)$$

where $k=1, 2, 3, \ldots$

346. It is obvious that the first term a and the common difference d of the progression can be assumed to be relatively prime numbers because if both a and d were divisible by a number $k > 1$ then we could simply cancel by k all the terms of the progression. Further, by virtue of Euler's theorem (see Problem 341), for relatively prime a and d there is an integer r such that the number a^r-1 and, together with it, $a^{r+1}=a+Nd$, are divisible by d. Therefore $a^{r+1}-a=Nd$, that is $a^{r+1}=a+Nd$ where N is a natural number, whence it follows that the number a^{r+1} belongs to the given arithmetic progression. Moreover, in this case all the numbers a^{rk+1} where $k=1, 2, 3, \ldots$ also belong to the progression since the number $a^{rk}-1=(a^r-1)(a^{r(k-1)}+a^{r(k-2)}+\ldots+1)$ is divisible by a^r-1 and, consequently, it is divisible by d, whence it follows that the number $a^{r(k+1)}-a=Md$ (where M is a natural number) is divisible by d, and the number $a^{rk+1}=a+Md$ is the $(M+1)$th term of the progression. It is clear that the numbers $a, a^{r+1}, a^{2r+1}, a^{3r+1}, \ldots$ contain powers of the same prime factors (the exponents of the powers are different), which completes the proof of the assertion stated in the problem.

347. Let a be one of the numbers belonging to the sequence $2, 3, \ldots, p-2$. We shall consider the numbers

$$a, \; 2a, \; \ldots, \; (p-1)a$$

Among them there are not two numbers whose division by p leaves equal remainders. Consequently, the remainders resulting from their division by p are $1, 2, \ldots, p-1$, each of the remainders occurring only once (cf. the solution of Problem 340). In particular, in the sequence $1, 2, \ldots, p-1$ there is an integer b such that the division of ba by p leaves a remainder of 1. For this number b we must have $b \neq 1$ and $b \neq p-1$ because $2 \leqslant a \leqslant p-2$ and, consequently, for $b=1$ the division of the number $ba=a$ by p leaves the remainder $a \neq 1$ and for $b=p-1$ the division of the number $ba=(p-1)a=pa-a$ by p leaves the remainder $p-a \neq 1$. Besides, we have $b \neq a$ because if the division of a^2 by p left the remainder 1 then the number $a^2-1=(a+1)(a-1)$ would be divisible by p, which is only possible when $a=1$ and $a=p-1$. Consequently, $2 \leqslant b \leqslant p-2$ and $b \neq a$, that is the members of the sequence $2, 3, \ldots, p-2$ split into pairs of numbers the division of whose products by p leaves the remainder 1.

The product $2 \cdot 3 \cdot \ldots \cdot (p-2)$ contains $(p-3)/2$ such pairs of numbers, and the remainder resulting from the division of this product by p is also equal to 1. Further, the division of the number $p-1$ by p gives -1 in the remainder. Consequently, the division of the number $(p-1)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) = [2 \cdot 3 \cdot \ldots \cdot (p-2)] \cdot (p-1)$ by p leaves the remainder -1, that is $(p-1)! = kp-1$, whence $(p-1)!+1=kp$. Thus, $(p-1)! + 1$ is divisible by p.

If the number p is not prime, it has a prime divisor $q<p$. In this case $(p-1)!$ is divisible by q; therefore $(p-1)!+1$ is not divisible by q, and hence it cannot be divisible by p either.

348. (a) For $p=2$ we can write $p=1^2+0^2+1$. Now let the prime number p be odd; we shall show that in this case there are two numbers x and y which are *both less than* $p/2$ and satisfy the condition of the problem.

Let us consider the sequence consisting of the $(p+1)/2$ numbers $0, 1, 2, \ldots, (p-1)/2$. The division of the squares of any two of these numbers by p leaves different remainders; for, if we had

$$x_1^2 = k_1p + r \quad \text{and} \quad x_2^2 = k_2p + r$$

where x_1 and x_2 belong to that sequence then the equality

$$x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = (k_1 - k_2)p$$

would hold, that is the number $(x_1 - x_2)(x_1 + x_2)$ would be divisible by p, which is impossible since $x_1 < p/2$, $x_2 < p/2$, $x_1 + x_2 < p$ and $|x_1 - x_2| < p$ (we remind the reader that p is a prime number). Thus, the $(p+1)/2$ numbers

$$0^2, \quad 1^2, \quad 2^2, \quad \ldots, \quad \left(\frac{p-1}{2}\right)^2$$

give $(p+1)/2$ different remainders when they are divided by p. It follows that the division by p of the $(p+1)/2$ (negative) numbers $-1, -1^2-1, -2^2-1, \ldots, -[(p-1)/2]^2-1$ also leaves $(p+1)/2$ different remainders (if the remainders resulting from the division of $-x_1^2-1$ and $-x_2^2-1$ were the same then the division of x_1^2 and x_2^2 would also leave equal remainders)*. Since the division by p can result in only p different remainders (namely, 0, 1, 2, ..., $p-1$), it is clear that among the $p+1$ numbers $0^2, 1^2, 2^2, \ldots, [(p-1)/2]^2, -1, -1^2, -1, -2^2-1, \ldots$ $\ldots, -[(p-1)/2]^2-1$ there are at least two whose division by p leaves equal remainders. According to what was proved above, one of the numbers belonging to such a pair must necessarily be of the form x^2 and the other of the form $-y^2-1$. Now, if $x^2 = kp + r$ and $-y^2-1 = lp + r$ then

$$x^2 + y^2 = (k-l)\,p - 1 = mp - 1$$

whence it follows that $x^2+y^2+1 = mp$ is divisible by p.

Remark. In the condition of the problem we can additionally require that the two sought-for numbers x and y should not exceed $p/2$, that is we are allowed to impose the condition that the sum x^2+y^2+1 must be less than p^2; under this condition the quotient m resulting from the division of the sum x^2+y^2+1 by p is less than p.

(b) Let $p = 4n+1$ be a prime number. By virtue of Wilson's theorem (see Problem 347), the number

$$(p-1)!+1 = 1\cdot 2\cdot 3\cdot \ldots \cdot (4n)+1$$

is divisible by p. Now let us replace all those factors in the last expression which exceed $(p-1)/2 = 2n$ by the corresponding differences between the number p and numbers smaller than $(p-1)/2$ (these differences are equal to the factors they replace):

$$(p-1)!+1 = 1\cdot 2\cdot 3\cdot \ldots \cdot 2n\,(p-2n)\,(p-2n+1)\cdot \ldots$$
$$\ldots \cdot (p-1)+1 = (1\cdot 2\cdot 3\cdot \ldots \cdot 2n)\left[Ap + (-1)^{2n}\,2n\cdot\right.$$
$$\left.\cdot (2n-1)\cdot \ldots \cdot 1\right]+1 = A_1 p + (1\cdot 2\cdot 3\cdot \ldots \cdot 2n)^2+1$$

Since this number is divisible by p, the sum $((2n)!)^2+1$ is also divisible by p. Thus, the condition of the problem is satisfied by the number $x = (2n)! = [(p-1)/2]!$.

Remark. It should be noted that if the division of the number x by p leaves a remainder x_1 then the divisibility of the number $x^2+1 = (kp+x_1)^2+1 =$

* The quotient k and the remainder r resulting from the division of an integer a by p are determined by the formula $a = kp + r$ where $0 \leqslant r < p$ (in case a is negative the quotient k is also negative).

$= (k^2p + 2kx_1)p + x + 1$ by p implies that the number $x_1^2 + 1$ is also divisible by p. This allows us to assume that the number x mentioned in the condition of the problem is less than p, the number $x^2 + 1$ is less than p^2 and the quotient m resulting from the division of $x^2 + 1$ by p is less than p.

349. The existence of an infinitude of prime numbers follows from the result of Problem 234. (This problem also shows that in the sequence of all natural numbers the prime numbers occur "sufficiently often", for instance, "more often" than the perfect squares; see the remark to that problem.) From the result of Problem 90 it can also be seen that there exist infinitely many prime numbers: if the total number of prime numbers were $n < \infty$ then there could not exist more than n pairwise relatively prime integers. However, the following proof of the existence of an infinitude of prime numbers suggested by Euclid is perhaps the simplest.

Let us suppose that there are only n prime numbers 2, 3, 5, 7, 11, ..., p_n; then the number $N = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot \ldots \cdot p_n + 1$ exceeds all the prime numbers 2, 3, 5, ..., p_n and therefore N must be a composite number. However, the number $N - 1$ is divisible by 2, 3, 5, 7, ..., p_n, and therefore N must be relatively prime to *all* the prime numbers. We have thus arrived at a contradiction, which proves the theorem.

350. **(a)** The proof of this theorem is rather close to Euclid's proof of the existence of infinitely many prime numbers.

Let us suppose that among the numbers of the form $4k - 1$ there is only a finite set of prime numbers, namely 3, 7, 11, 19, 23, ..., p_n. Let us form the number $N = 4(3\cdot 7\cdot 11\cdot 19\cdot 23\cdot \ldots \cdot p_n) - 1$. It is greater than all the prime numbers belonging to the progression under consideration and hence it must be composite. Let us express N as a product of prime factors. Among these factors there cannot be numbers of the form $4k - 1$ because the number $N + 1 = 4(3\cdot 7\cdot 11\cdot 19\cdot 23\cdot \ldots \cdot p_n)$ is divisible by *all* prime numbers of the form $4k - 1$ and, consequently, the number N is relatively prime to all these numbers. Since the number N is odd it must be equal to a product of several prime numbers of the form $4k + 1$. But this is impossible. Indeed, a product of two numbers of the form $4k + 1$ has the same form:

$$(4k_1 + 1)(4k_2 + 1) = 16k_1k_2 + 4k_1 + 4k_2 + 1 = 4(4k_1k_2 + k_1 + k_2) + 1 = 4k_3 + 1$$

Consequently, a product of several numbers of the form $4k + 1$ also has the same form whereas the number N has the form $4k - 1$. The contradiction we have arrived at proves the theorem.

It can similarly be proved that among the members of the progression 5, 11, 17, 23, ... there are also infinitely many prime numbers (these are prime numbers of the form $6k - 1$).

(b) The proof of this theorem is based on the same idea as the one used in the proofs of the theorems of Problem 350 (a) but is a little more complicated.

Let us suppose that among the numbers belonging to the sequence 11, 21, 31, 41, 51, 61, ... there are only a finite number of prime numbers: 11, 31, 41, 61, ..., p_n. Let us form the number $N=(11\cdot 31\cdot 41\cdot 61\cdot \ldots \cdot p_n)^5-1$. It is relatively prime to all prime numbers 11, 31, 41, ..., p_n because the number $N+1$ is divisible by all these numbers. On denoting by a the product $11\cdot 31\cdot 41 \ldots p_n$ we can write $N=a^5-1=(a-1)(a^4+a^3+ a^2+a+1)$.

Let us investigate the prime divisors of the second factor $a^4+ a^3+a^2+a+1$ in the last product. It is obvious that the sum $a^4+a^3+a^2+a+1$ is not divisible by 2 (because a sum of five odd numbers is itself odd). Further, the number $a^4+a^3+a^2+ a+1$ is divisible by 5 since a ends with 1 (because a is equal to a product of a number of factors each of which ends with 1), the numbers a^2, a^3 and a^4 all have 1 at their end, and, consequently, the sum $a^4+a^3+a^2+a+1$ ends with 5. Now let p be a prime divisor of the number $a^4+a^3+a^2+a+1$ different from 5. Then $a-1$ cannot be divisible by p because, if otherwise, the number a would be of the form $kp+1$ and, consequently, the numbers a^2, a^3 and a^4 (they are equal to $(kp+1)^2$, $(kp+1)^3$ and $(kp+1)^4$ respectively) would have that same form and therefore the division of the number

$$a^4+a^3+a^2+a+1=(kp+1)^4+(kp+1)^3+(kp+1)^2+(kp+1)+1$$

by p would leave the remainder 5. It follows that $p-1$ must be divisible by 5. Indeed, for instance, let us suppose that the remainder resulting from the division of $p-1$ by 5 is equal to 4, that is $p-1=5k+4$. It should be noted that, by Fermat's theorem (see Problem 240), the difference $a^{p-1}-1$ is divisible by p, and therefore in the case under consideration we must have

$$a^{p-1}-1=a^{5k+4}-1=a^4(a^{5k}-1)+(a^4-1)$$

Further, since $a^{5k}-1=(a^5)^k-1^k$ is divisible by a^5-1 and therefore by p as well, the difference a^4-1 is also divisible by p. But we have $a^5-1=a(a^4-1)+(a-1)$, and, consequently, if a^5-1 and a^4-1 were divisible by p, the difference $a-1$ would also be divisible by p, which, as was shown above, is impossible. The fact that the remainder resulting from the division of the number $p-1$ by 5 cannot be equal to 1, 2 or 3 is proved similarly.

Thus, the number $p-1$ is divisible by 5 and is even ($p-1$ is even because p is odd). Consequently, $p-1$ is divisible by 10; therefore the number p has the form $10k+1$ and hence it belongs to the progression under consideration. We have thus shown that the prime divisors of the sum $a^4 + a^3 + a^2 + a + 1$ can only be the number 5 and prime numbers of the form $10k + 1$.

Further, the number $a^4 + a^3 + a^2 + a + 1$ is obviously greater than 5 and is not divisible by $5^2 = 25$. Indeed, the number a ends with 1 and, consequently, has the form $5k + 1$. By Newton's binomial formula, we have

$$a^4 + a^3 + a^2 + a + 1 = (5k + 1)^4 + (5k + 1)^3 + (5k + 1)^2 + \\ + 5k + 1 + 1 = 625k^4 + 4 \cdot 125k^3 + 6 \cdot 25k^2 + 4 \cdot 5k + \\ + 1 + 125k^3 + 3 \cdot 25k^2 + 3 \cdot 5k + 1 + 25k^2 + 2 \cdot 5k + 1 + \\ + 5k + 1 + 1 = 625k^4 + 5 \cdot 125k^3 + 10 \cdot 25k^2 + 10 \cdot 5k + 5 = \\ = 5 \cdot [5(25k^4 + 25k^3 + 10k^2 + 2k) + 1]$$

It follows that this sum and, consequently, the number $N = a^5 - 1$ must have at least one prime divisor of the form $10k + 1$. At the same time, by the hypothesis, the number N is relatively prime to *all* prime numbers of the form $10k + 1$; we have thus arrived at a contradiction, which proves the theorem.

Remark. It should be noted that the proof presented here can be applied almost without any changes in order to show that every arithmetic progression composed of the numbers of the form $2pk + 1$ where p is an arbitrary odd prime number contains infinitely many prime numbers.

Answers and Hints

1. The tallest of the smallest.

2. Consider the sum of the numbers of times each person has ever shaken hands with other people.

3. Let A be one of the six people; this person either has three acquaintances or there are three people with whom A is not acquainted.

4. **(a)** It is impossible. **(b)** Construct an example satisfying the conditions of the problem.

5. Prove that if A and B are not acquainted then they have two mutual acquaintances.

6. Consider the scientist who has the *greatest* number of acquaintances among the participants.

7. Exclude consecutively from the delegates the pairs of delegates speaking one language.

8. Let A be an arbitrary participant of the conference; show that there is the language in which he can speak with not less than 6 other participants.

9. $n = k(k+1)/2 + 1$ where k is an integer.

10. 5000 days (in the town there are two parties such that two inhabitants are friends if and only if they belong to one party).

11. If the knight travels sufficiently long then there must be a part AB of his path (where A and B are castles) along which (in the direction from A to B) the knight goes not less than three times.

12. Let A and B be two enemies sitting next to each other; prove that Merlin can make a part of the knights change their places so that the pair of enemies A, B sitting next to each other is replaced by a pair A, A' of friends sitting next to each other while none of the pairs of friends sitting next to each other is replaced by a pair of enemies.

13. **(a)** In the first weighing place 27 coins on each of the scale pans. **(b)** The number k is determined by the inequalities $3^{k-1} < n \leqslant 3^k$.

14. First put one cube on each scale pan; then put both these cubes on one scale pan and then, in succession, put all the possible pairs of the remaining cubes on the other scale pan.

15. In the first weighing put four coins on each scale pan.

16. **(a)** One link. **(b)** Seven links.

17. Let S be one of the underground stations; consider a station T which is the farthest from S.

18-19. Use the method of mathematical induction.

20. Use the proof by contradiction. To this end assume that the assertion of the problem is false and show that under this assumption there is an infinite number of towns in the state of Shvambrania (when constructing this infinite sequence of towns it is advisable to use the method of mathematical induction).

21. It cannot.

22. It is sufficient for the king to move first to one of the corners of the chess-board and then along the diagonal of the chess-board.

23. Change the order of the arrangement of the squares so that it becomes possible to move from any square to the neighbouring ones.

24. Prove (say, using the induction method) that if in a group of students exactly n people speak each of the three languages (where $n \geqslant 2$) then it is possible to form a subgroup in which exactly 2 students speak each of the languages.

25. (a) $2\frac{2}{3}$. (b) 20.
26. 6 days; 36 sets of medals.
27. 15 621.
28. Two rubles.
29. (a) In the Gregorian Calendar (which is in general use) every year, except the leap years, has 365 days. Each leap year has an additional day (the 29th of February). The leap years are those whose numbers are divisible by 4 except the years divisible by 100 but not divisible by 400. It follows that every 400 years contain an integral number of weeks; consequently, it only remains to check what day of the week, Saturday or Sunday, is more frequently the New Year day. *Answer*: Sunday. (b) Friday.
30. All the numbers ending with 0 and the two-digit numbers 11, 23, 33, 44, 55, 66, 77, 88, 99; 12, 24, 36, 48; 13, 26, 39; 14, 28; 15; 16; 17; 18; 19.
31. (a) $625\underbrace{0\ldots0}_{n\text{ times}}$; $n = 0, 1, 2, \ldots$. (b) Try to solve the following problem: find a whole number starting with a known digit a which decreases 35 times when this digit is deleted.
32. (a) Start with proving the auxiliary proposition: the number in question decreases 9 times when the second (counting from right to left) digit 0 is deleted. (b) 10 125 2025; 30 375; 405; 50 625; 6075, 70 875 (at the end of each of these numbers an arbitrary number of zeros can be additionally written).
33. (a) The numbers whose all digits except the first two are zeros. (b) Investigate separately the cases when the first digit of the sought-for number is 1, 2, 3, ..., 9. There are altogether 104 different numbers satisfying the condition of the problem at the end of each of which an arbitrary number of zeros can be additionally written.
34. (a) The smallest possible number is 142 857. (b) The digit 1 or 2. The smallest of the numbers with initial digit 2 in 285 714.
35. 153 846.
36. Use the property that the numbers divisible by 5 must end with the digit 0 or 5; the numbers divisible by 6 or by 8 end with even digits.
37. Try to solve the following problem; find the number which increases twice when its initial digit is carried to the end.
38. The problem is solved by analogy with the preceding one.
39. The smallest number satisfying the condition of the problem is 7 241 379 310 344 827 586 206 896 551.
40. (a) A number which is 5, 6, 8 or 7 times as small as its reversion must begin with the digit 1; a number which is twice or three times as small as its reversion can begin with the digits 1, 2, 3, 4 or 1, 2, 3 respectively. (b) The numbers that are 4 times as small as their reversions are

$$0;\quad 2\,178;\quad 21\,978;\quad 219\,978;\quad 2\,199\,978;\ \ldots \qquad (*)$$

and also the numbers with decimal representation of the form $P_1P_2\ldots$ $\ldots P_{n-1}P_nP_{n-1}\ldots P_2P_1$ where P_1, $P_2\ldots P_n$ are some numbers belonging to sequence (*).
41. (a) 142 857. (b) Try to find an 8-digit number which increases 6 times when its last four digits are carried to the beginning while their order is preserved.
42. 142 857.
43. 111, 222, 333, ..., 999, 407, 518, 629, 370, 481, 592.
44-45. Consider the process of the addition of the given numbers written in a column.
46. Factor the polynomials indicated in the condition of the problem; find what remainders can result from the division of the number n by 3 (accordingly by 5, by 7 etc.).

47. **(a)** Use the property that a difference of two powers with equal even exponents is divisible by the sum of the bases of the powers. **(b)** See the hint to Problem 46.

48. **(a)** $56\,786\,730 = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot31\cdot61$. Use the propositions established in Problems 46 (a)-(e) and similar propositions implied by Fermat's theorem (see Problem 340). **(b)** Consider the factorization of the given expression and compare the number of the factors with that of the factorization of the number 33. **(c)** Make use of the identity $n^2 + 3n + 5 = (n + 7)(n - 4) + 33$.

49. For even n.

50. It does not exist.

51. Take into account that every integer not divisible by 5 can be written in the form $5k \pm 1$ or $5k \pm 2$. *Answer*: 0 or 1.

52. Make use of the result of the preceding problem.

53. 625 or 376.

54. Determine the last two digits of the number N^{20} and the last three digits of the number N^{200}. *Answer*: 7; 3.

55. $1 + 2 + 3 + \ldots + n = n(n + 1)/2$. Grouping some terms of the sum $1^k + 2^k + 3^k + \ldots + n^k$ prove that the sum is divisible by $n/2$ and by $n + 1$ or by n and by $(n + 1)/2$.

56. The difference between the sum of the digits of the number occupying even places and the sum of the digits occupying odd places must be divisible by 11.

57. The number is divisible by 7.

58. It is always possible to find a number starting with the digits, 1, 0 which is divisible by K. It is possible to prove that 9 is divisible by K by performing an appropriate permutation of the digits of the above-mentioned number divisible by K and subtracting from each other two numbers divisible by K.

59. The sought-for number consists of 300 ones.

60. Investigate the last digits of the numbers of the form $N = 2^k$ (where $k = 1, 2, 3, \ldots$) and also consider the remainders resulting from the division of the numbers N by 3.

61. $26\,460 = 2^2\cdot3^3\cdot5\cdot7^2$. Prove separately that the given expression is divisible by $5\cdot7^2$ and that it is divisible by $2^2\cdot3^3$.

62. Use the equality

$$11^{10} - 1^{10} = (11 - 1)(11^9 + 11^8 + 11^7 + 11^6 + 11^5 + 11^4 + 11^3 + 11^2 + 11 + 1)$$

63. Write the given number in the form

$$(2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$$

64. Use the method of mathematical induction.

65. Use the fact that $10^6 - 1 = 999\,999$ is divisible by 7 and that the division of any power of ten by 6 leaves a remainder of 4. *Answer*: 5.

66. **(a)** 9; 2. **(b)** 88; 67. **(c)** Find the last two digits of the numbers $7^{14^{14}}$ and $2^{14^{14}}$. *Answer*: 36.

67. **(a)** Both numbers have the digits 89 at the end. **(b)** Prove that the difference of the given numbers is divisible by $1\,000\,000 = 2^6\cdot5^6$.

68. **(a)** 7; 07. **(b)** 3; 43.

69. Consider the numbers

$$Z_1 = 9, \quad Z_2 = 9^{Z_1}, \quad Z_3 = 9^{Z_2}, \quad \ldots, \quad Z_{1001} = 9^{Z_{1000}} = N$$

and determine consecutively the last digit of the number Z_1, the last two digits of the number Z_2, the last three digits of the number Z_3, the last four digits of the number Z_4, the last five digits of the number Z_5 and the last five digits of the numbers $Z_6, Z_7, \ldots, Z_{1001} = N$. *Answer*: 45 289.

70. Compile the tables of the remainders resulting from the division of the numbers 5^n and n^5 by the number 13. The smallest number n satisfying the condition of the problem is $n = 12$.

71. For all a multiple of 4 the last two digits of the number under consideration are 30.

72. The sought-for 1000 digits can be written as a sequence of the form $p\underbrace{PP\ldots P}_{23 \text{ times}}$ where

$$P = 0204081632653061224489795918367346938775\,51$$

Here P is the period of the periodic fraction to which 1/49 is changed and p is the group of the last 34 digits of the number P. To elaborate the proof make use of the obvious equality

$$N = \frac{50^{1000} - 1}{50 - 1} = \frac{50^{1000} - 1}{49}$$

73. Consider the difference $M - 3N$.

74. 24.

75. **(a)** Compare the exponents of the powers of a prime number p which are contained in $a!$ and in the product $(t+1)(t+2)\ldots(t+a)$. **(b)** and **(c)** Use the result of Problem 71 (a). **(d)** First prove that there exists a number k such that the division by $n!$ of the product kd where d is the common difference of the progression leaves a remainder of 1.

76. It is not divisible by 7.

77. **(a)** The number $(n-1)!$ is not divisible by n when n is a prime number and when $n = 4$. **(b)** The number $(n-1)!$ is not divisible by n^2 when n is a prime number or a duplicated prime number or is equal to 8 or is equal to 9.

78. Prove that all such numbers are less than $7^2 = 49$. *Answer*: 24, 12, 8, 6, 4 and 2.

79. **(a)** Prove that a sum of squares of five consecutive whole numbers is divisible by 5 and is not divisible by 25.

(b) Find the remainder resulting from the division of a sum of even powers of three consecutive whole numbers by 3.

(c) Determine the remainder resulting from the division by 9 of a sum of powers with equal even exponents of nine consecutive whole numbers.

80. **(a)** Find what remainders result from the division of the numbers A and B by 9. **(b)** 192, 384, 576, or 273, 546, 819, or 327, 654, 981 or 219, 438, 657.

81. These digits are four noughts.

82. Use Pythagoras' theorem.

83. Consider the remainder resulting from the division of the expression $b^2 - 4ac$ by 8.

84. Prove that after the fractions are added together and the sum is cancelled (if possible) the denominator of the resulting fraction is divisible both by 3 and by 2.

85. To prove that M and N are not integral numbers it is required to show that after the addition we obtain a fraction whose denominator is divisible by a power of 2 higher than that by which the numerator is divisible.

When proving that K is not an integral number we should replace in the preceding argument powers of two by powers of three.

86. **(a)** Use the fact that the fractions p/q and q/p are simultaneously reducible or irreducible. **(b)** It can be reduced by 13.

87. Let $N = a \cdot 10^{1952} + A$ be one of the numbers which are read as indicated in the condition of the problem (where a is the first digit of the number N); prove that if N is divisible by 27 then the number $N_1 = 10A + a$ is also divisible by 27.

88. Prove that if the decimal representation of the number $a = 5^{1000}$ involves zeros then there is a number divisible by a the first zero in whose decimal representation (provided there exists such) is placed farther from the (right) end of that number than the first zero in the representation of a.

89. Start with proving the equality

$$(1+10^4+10^8+\dots+10^{4k})\cdot 101=(1+10^2+\dots+10^{2k})(10^{2k+2}+1)$$

90. Show that $(2^{2^n}+1)-2$ is divisible by all the preceding numbers of the given sequence; this will imply that $2^{2^n}+1$ and any of the preceding numbers in the sequence cannot have common divisors other than 2.

91. Consider the remainders resulting from the division by 3 of the numbers 2^n-1 and 2^n+1.

92. **(a)** Consider the remainders resulting from the division by 3 of the numbers p, $8p-1$, and $8p+1$. **(b)** Consider the remainders resulting from the division of the numbers p, $8p^2+1$ and $8p^2-1$ by the number 3.

93. Investigate the remainders obtained in the division of a prime number by 6.

94. See the hint to the foregoing problem.

95. **(a)** Prove that the common difference of the progression must be divisible by $2\cdot3\cdot5\cdot7=210$. *Answer*: 199, 409, 619, ..., 2089. **(b)** Prove that if the first term of the progression is different from 11 then the common difference must be divisible by $2\cdot3\cdot5\cdot7\cdot11=2310$; if the first term of the progression is equal to 11 then the common difference must be divisible by 210 (In the solution of Problems 95 (a) and (b) it is advisable to use table of prime numbers.)

96. **(a)** Such is an odd number not divisible by 3.

(b) It is sufficient to find a number among the given 16 numbers which does not have common divisors equal to 2, 3, 5, 7, 11 or 13 with the other 15 numbers.

97. The product is equal to $\underbrace{22\ldots\ldots2}_{665\text{ times}}1\underbrace{77\ldots\ldots7}_{665\text{ times}}8$.

98. The quotient is equal to $\underbrace{777\,000\,777\,000\ldots777\,000}_{\text{the combination }777\,000\text{ is repeated }166\text{ times}}77$; the remainder is equal to 700.

99. $222\,222\,674\,025=471\,405^2$.

100. They do not exist.

101. 523 152 and 523 656.

102. 1946.

103. **(a)** Transform the indicated number and compare it with the expression for the sum of terms of an arithmetic progression with common difference 1 whose first term is 10^{n-1} and last term 10^n. **(b)** 1 769580.

104. Begin with considering all whole numbers from 0 to 99 999 999; at the left end of those of them which consist of less than eight digits write additionally a number of zeros so that they all become 8-digit expression.

105. 7.

106. No.

107. The number of ones exceeds by unity that of twos.

108. It cannot.

109. This number is divisible by 11 111.

110. 6 210 001 000.

111. Since the given number A is equal to 10^9-1, for any number $X=\overline{x_1x_2\ldots x_k}$, we have $AX=\overline{x_1x_2\ldots x_k000000000}-\overline{x_1x_2\ldots x_k}=M-N$. On writing the numbers M and N in a column consider the process of subtraction of N from M.

112. The condition of the problem is satisfied by all numbers $N\geqslant A$ such that $N=10^m-1$.

113. Use the induction method (with respect to the number n); for $m \geqslant n$ the assertion of the problem is false.

114. It suffices to note that among the first four numbers of every row there is an even number.

115. Prove that the sum of all numbers in every row of the table (beginning with the second one) is divisible by 1958.

116. 40.

117. On denoting by $12 + x$ the time of the beginning of the first performance and by y the duration of the performance we can readily set a linear system of inequalities for the numbers x and y.

118. The possible values of T (expressed in minutes) are $20, 15, 12, 7\frac{1}{2}$ and $5\frac{5}{11}$.

119. 100.

120. **(a)** The sought-for number must begin with the greatest possible number of nines. **(b)** The answer is the same with nines replaced by zeros.

121. **(a)** 147; 258; 369. **(b)** 941, 852; 763.

122-123. Apply the formula for the sum of the terms of an arithmetic progression.

124. Write the expression $n(n+1)(n+2)(n+3)+1$ in the form of a square of a polynomial.

125. Prove that among the numbers in question there cannot be more than four pairwise distinct.

126. Divide 9 weights with consecutively increasing magnitudes into three groups two of which are of the same weight while the third one is lighter.

127. Prove that the numbers of grams the given weights weigh are all either even or odd.

128. It suffices to consider the case when all the numbers in question are positive and their product is equal to 1.

129. After 2^k operations we inevitably arrive at an N-tuple of ones.

130. Prove that in the transformation process described in the condition of the problem the differences between the given numbers permanently decrease.

131. $(x, y, z) = (1, 1, 0)$.

132. **(a)** First of all prove that for any original numbers we eventually arrive at a 4-tuple of even numbers. **(b)** The assertion stated in Problem 132 (a) remains true for rational numbers and is false for irrational numbers (in the latter case the original numbers can be chosen so that all the following 4-tuples are proportional to the first one).

133. **(b)** Construct an increasing sequence beginning with the first of the given 101 numbers. If this sequence contains less than 11 numbers, delete these numbers from the original set and construct a new increasing sequence beginning with the first of the remaining numbers; if that sequence again contains less than 11 numbers, delete these numbers as well and construct a new increasing sequence, and so on. If all the sequences thus constructed contain less than 11 numbers each, then the total number of these sequences is not less than 11; using this fact we can construct a decreasing sequence of 11 numbers.

134. Consider the greatest odd divisors of the given numbers.

135. **(a)** Consider the remainders with the smallest absolute values resulting from the division of the numbers by 100.

(b) Let $a_1, a_2, \ldots, a_{100}$ be the given numbers. Consider the remainders resulting from the division by 100 of the numbers a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, ...

(c) If the sum of several numbers is less than 200 and is divisible by 100 then it is equal to 100.

(d) Prove in succession the following properties: *among any* 3 *integral numbers there are* 2 *numbers whose sum is divisible by* 2; *among any* 9 *integral numbers there are* 5 *numbers whose sum is divisible by* 5; *among any* 199 *inte-*

gral numbers there are 100 *numbers whose sum is divisible by* 100 (when proving the third property use the first two).

136. Elaborate the proof by contradiction.

137. Consider the numbers of passages from a cross to a nought and from a nought to a cross encountered when the circle is described in one chosen direction.

138. Consider the sum of all the factors of this product.

139. One half of the summands in the given sum consists of the numbers $+1$ and the other half of the numbers -1.

140. Collect in the first group all numbers whose decimal representations contain an even number of ones.

141. Write the 5 numbers one below another; then the number of the columns of digits containing two identical digits lies within the limits from 400 to 600.

142. Prove that after a number of operations have been performed we can always change any sign while the other signs remain unchanged.

143. Use Dirichlet's principle (see page 9).

144. Change the number $1/N$ to a (periodic) decimal.

145. Let d be the common difference of an arithmetic progression and let $\alpha = \{d\}$ (see page 36) for a nonintegral d and $\alpha = 1$ for an integral d. It is sufficient to place the line segments of length 1 in such a way that a line segment of length α cannot be formed of intervals between the former segments or of its parts (here are meant the intervals between the line segments or its parts lying "sufficiently far" from the point representing the first term of the progression).

146. Prove that for $A \leqslant m + n$ the interval $(0, A)$ of the number line contains exactly $A - 1$ of the given fractions.

147. Denote by k_i $(i = 1, 2, 3, \ldots)$ the number of those members of the given sequence of positive integers which lie between $1000/i$ and $1000/(i+1)$ and compute the number of the numbers less than 1000 which are multiple of at least one of the numbers $a_1, a_2, \ldots, a_n$.

148. The length k of the period of p/q is equal to the smallest exponent k for which $10^k - 1$ is divisible by q. If $k = 2l$ then it follows that $10^l - 1$ is divisible by q, that is $(10^l + 1)/q$ is an integral number. The last fact implies that for the period $\overline{a_1a_2 \ldots a_la_{l+1}a_{l+2} \ldots a_k}$ of the fraction p/q we have

$$a_1 + a_{l+1} = a_2 + a_{l+2} = \ldots = a_1 + a_2 = 9$$

149. Use the fact that the number of digits in the periods of the fractions a_n/p^n and a_{n+1}/p^{n+1} are equal to the smallest positive numbers k and l respectively such that $10^k - 1$ is divisible by p^n and $10^l - 1$ is divisible by p^{n+1}.

150. **(a)** 7744. **(b)** 29; 38; 47; 56; 65; 74; 83 and 92.

151. Let a denote the number formed of the first two digits of the sought-for number and let b denote the number formed of the last two digits; then $99a = (a+b)^2 - (a+b) = (a+b)(a+b-1)$. *Answer*: 9801; 3025; 2025.

152. **(a)** 4624; 6084; 6400; 8464. **(b)** Such numbers do not exist at all.

153. **(a)** 145. **(b)** Only the number 1.

154. **(a)** 1; 81. **(b)** 1; 8; 17; 18; 26; 27.

155. **(a)** The number x cannot exceed 4. *Answer*: $x = 1$, $y = \pm 1$; $x = 3$, $y = \pm 3$. **(b)** $x = 1$, $y = \pm 1$, z is an arbitrary even number; $x = 3$, $y = \pm 3$, $z = 2$; $x = 1$, $y = 1$, z is an arbitrary odd number; x is an arbitrary positive integer, $y = 1! + 2! + \ldots + x!$, $z = 1$.

156. Consider the exponents of the powers of two by which the sought-for four numbers can be divisible. *Answer*: the expansion is impossible for an odd n and there exists only one expansion for an even n:

$$2^n = \left(2^{\frac{n}{2}-1}\right)^2 + \left(2^{\frac{n}{2}-1}\right)^2 + \left(2^{\frac{n}{2}-1}\right)^2 + \left(2^{\frac{n}{2}-1}\right)^2$$

157. See the hint to Problem 156. Problem 157 (b) is similar to Problem 157 (a): there exists only one answer, namely $x = y = z = v = 0$.

158. (a) It can be shown that if the numbers x, y and z satisfy the indicated equality and, for instance, the inequality $z > kxy/2$ holds, then the numbers can be decreased in such a way that the same equality remains valid for them. In case $x \leqslant kyz/2$, $y \leqslant kxz/2$, $z \leqslant kxz/2$ and $x \leqslant y \leqslant z$ there must be $2 \leqslant kx \leqslant \leqslant 3$. *Answer*: $k = 1$ and $k = 3$. **(b)** Every such triple of integers can be obtained with the aid of a number of consecutive substitutions of the form $x_1 = x$, $y_1 = y$, $z_1 = kxy - z$ from one of the triples 1, 1, 1 and 3, 3, 3. Altogether, among the first 1000 numbers there are 23 triples of numbers satisfying the conditions of the problem.

159. Prove that x, y and z, are even. *Answer*: $x = y = z = 0$.

160. $(x, y) = (0, -1), (-1, -1), (0, 0), (-1, 0), (5, 2), (-6, 2)$.

161. $x = 3$, $y = 1$.

162. $x = n^2$, $y = 1$, $z = n$ or $x = 0$, $y = m$, $z = 0$.

163. Suppose that the assertion stated in the condition of the problem is false and consider the greatest prime number for which there are solutions.

164. Consider in succession the following cases: all the four numbers are distinct, two of the numbers coincide while the other two are distinct, there are two pairs of pairwise equal numbers etc. *Answer*. 96, 96, 57, 40; 11, 11,6, 6; $k(3k \pm 2)$, $k(3k \pm 2)$, $k(3k \pm 2)$, 1 (here k is an arbitrary integer such that $k(3k \pm 2)$ is positive); 1, 1, 1, 1.

165. 2, 2 and 0, 0.

166. $1 = 1/2 + 1/4 + 1/4 = 1/2 + 1/3 + 1/6 = 1/3 + 1/3 + 1/3$.

167. (a) Put $x = x_1 + n$, $y = y_1 + n$. **(b)** Cf. the hint to Problem 167 (a). **(b)** $x = m(m + n)t$, $y = n(m + n)t$, $z = mnt$ where m, n and t are arbitrary integers.

168. (a) Let $y > x$; show that in this case y is divisible by x. *Answer*: $x = = 2$, $y = 4$. **(b)** $x = [(p + 1)/p]^p$, $y = [p + 1)/p]^{p+1}$ where p is an arbitrary integer different from 0 and -1.

169. 7 or 14.

170. From the relationship between the number of points received by the pupils of the 6th form and the number of games they played one can conclude that all the pupils of the 6th form won all the games they played. It follows that only one pupil of the 5th form participated in the tournament.

171. Denote $p - a = x$, $p - b = y$, $p - c = z$ where a, b and c are the sides of the triangle and p is half the perimeter of the triangle: $p = (a + b + c)/2$. Then the problem reduces to the determination of the integral solutions of the equation $xyz = 4(x + y + z)$ or of the equation $x = (4y + 4z)/(yz - 4)$. The condition $x \geqslant y$ can be regarded as a quadratic inequality with respect to y (with coefficients depending on z); this makes it possible to find the limits within which z and y must lie (altogether, there are 5 solutions to the problem).

172. $n(n^2 + 1)/2$.

173. Prove that every number occurs on the diagonal an odd number of times. For an even n the assertion of the problem is false.

174. The difference is equal to $n^2 - n$.

175. From the tables obtained as described choose the one with the *maximum* sum of all its numbers and investigate this table.

176. 1.

177. Use the property that the ith row, the $(9 - i)$th row, the ith column and the $(9 - i)$th column contain the same numbers.

178. Use the relation $a_{ij} + a_{kl} = a_{kj} + a_{il}$ which holds for all i, j, k and l.

179. Take into account that $a_{ii} = 0$ and $a_{il} = -a_{lj}$ for all i and j.

180. Use the method of mathematical induction.

181. (a) Consider the variation of the signs which stand in the 8 squares adjoining the edges of the board but are not at the corners. **(b)** Reduce the problem to Problem 180 (a).

182. **(a)** This is not always possible (prove that there are distributions of the signs that cannot be obtained from the one in which all the squares contain the sign "+" with the aid of the operations described in the condition of the problem). **(b)** This is not always possible(see what has been said in connection with Problem 182 (a)).

183. **(a)** Yes. **(b)** It is possible to transform consecutively into zero all the numbers in the 1st row, then the numbers in the 2nd row etc.

184. Write the number a in binary notation.

185. Make use of the induction method.

186. Let $u_{n+1} + u_{n+2} + \ldots + u_{n+8} = s_n$ (where u_k is the kth Fibonacci number); prove that $u_{n+9} < s_n < u_{n+10}$.

187. Consider the sequence of the remainders resulting from the division of the Fibonacci numbers by 5.

188. The last four digits of a difference of two numbers are completely determined by the last four digits of the minuend and of the subtrahend. Prove that there exist n and k such that the last four digits of the $(n+k)$th and of the $(n+k+1)$th Fibonacci numbers are equal to those of the kth and of the $(k+1)$th Fibonacci numbers respectively. This will mean that the last four digits of the $(n+k-1)$th and of the $(k-1)$th Fibonacci numbers coincide etc. In this way it is possible to find a Fibonacci number whose last four digits coincide with those of the first Fibonacci number which is equal to zero.

189. Use the inequalities $a_{n-1}^2 + 2 < a_n^2 < a_{n-1}^2 + 3$.

190. *First solution.* Add a number a_{n+1} to the sequence. *Second solution.* Apply the principle of mathematical induction.

191. 2952 (prove that the greatest number of members in a sequence satisfying the conditions of the problem which starts with the greatest number $a_1 = n$ is equal to $[(3n+1)/2]$).

192. Elaborate the proof by contradiction. Investigate the possible values of the digits α_n, α_{n+1}, ... such that when they are written additionally the given number always remains prime.

193. **(a)** This is impossible. **(b)** It is possible.

194. For the first time the number 81 occurs in the 111 111 111th place; the number 27 is consecutively repeated 4 times earlier than the number 36 first occurs.

195. 1972 times (take into account that if a and b are relatively prime numbers then the pair a, b occurs only once in the sequence of the collections I_0, I_1, ... and if a and b are not relatively prime then these numbers never occur).

196. Let α_4, α_3, α_2, α_1 be the last four digits of the sequence; determine the number of times the group of the digits $\alpha_1\alpha_2\alpha_3\alpha_4$ occurs in the given sequence.

197. Let N_k be the product of the first k prime numbers; prove (using the induction method) that the assertion stated in the condition of the problem holds for the numbers N_k with any k.

198. Prove that any natural number n can be written in a unique manner in the form $n = px + qy$ where x and y are integers and $0 \leqslant x < q$.

199. Represent the numbers indicated in the condition of the problem in the form $X(X+1)/2 + x$ where $X \geqslant 0$ and $0 \leqslant x \leqslant X$.

200. Consider all the points with integral coordinates (x, y) located within the square bounded by the coordinate axes and the straight lines $x = 100$ and $y = 100$.

201. Write x in the form $x = [x] + \alpha$ where $\alpha = \{x\}$.

202. Use the result of Problem 201 (3).

203. Consider all the points with integral coordinates x, y such that $0 < x < q$, $0 < y < p$ and $y/x \leqslant p/q$.

204. Apply the principle of mathematical induction. (It is also possible to solve the problem geometrically; to this end one should consider all the points with integral coordinates lying in the first quadrant below the hyperbola $xy = n$).

205-207. In the solution of Problem 205 one should use the equality

$$[(2+\sqrt{2})^n] = (2+\sqrt{2})^n + (2-\sqrt{2})^n - 1$$

Problems 206 (a), (b) and 207 are solved analogously.

208. Among the p consecutive whole numbers $n, n-1, n-2, \ldots, n-p+1$ there is one and only one number divisible by p. If this number is equal to N then $[n/p] = N/p$. Thus, the difference, $C(n, p) - [n/p]$ can be written in the form

$$\frac{n(n-1)\ldots(N+1)N(N-1)\ldots(n-p+1)}{p!} - \frac{N}{p}$$

209. If $\alpha > 0$ then $(N-1)/N \leqslant \alpha \leqslant N/(N-1)$.

210. Prove that $(N/2^k)$ is equal to the number of those whole numbers not exceeding N which are divisible by 2^{k-1} and are not divisible by 2^k.

211. 31.

212. **(a)** Compare the given product with the product $(2/3)\cdot(4/5)\cdot(6/7)\ldots$ $\ldots(98/99)$ or square the inequality that must be proved.

(b) Using the principle of mathematical induction prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\ldots\frac{2n-1}{2n} \leqslant \frac{1}{\sqrt{3n+1}}$$

213. The second number.

214. **(a)** The first of the numbers is smaller. **(b)** The first of the numbers is greater (make use of the method of mathematical induction).

215. Prove that if $10^{k-1} \leqslant 1974^n < 10^k$ then the inequality $1974^n + 2^n \geqslant 10^k$ cannot hold.

216. ± 11.

217. Use the inequality established in Problem 212 (a).

218. The number $99^n + 100^n$ is greater than 101^n for $n \leqslant 48$ and is smaller than 101^n for $n > 48$.

219. 300!

220. Begin with proving that

$$1+\frac{k}{n} \leqslant \left(1+\frac{1}{n}\right)^k \leqslant 1+\frac{k}{n}+\frac{k^2}{n^2}$$

for any positive integer $k \leqslant n$.

221-222. Use the result of Problem 220.

223. Use the method of mathematical induction.

224. Apply Newton's binomial formula.

225. Use the method of mathematical induction.

226. Use the inequality

$$(k+1)x^k(x-1) > x^{k+1} - 1 > (k+1)(x-1)$$

which implies that

$$(p+1)^{k+1} - p^{k+1} > (k+1)p^k > p^{k+1} - (p-1)^{k+1}$$

for any positive integer p.

227. These inequalities can be obtained by replacing the terms of the given sums by greater (accordingly, smaller) numbers; when necessary, group these terms in an appropriate manner before replacing the numbers (that is, when necessary, the indicated replacement should be performed for the new sums containing a smaller number of terms).

228. Begin with showing that

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

Answers: **(a)** 1998. **(b)** 1800.

229. Start with proving that

$$\frac{3}{2}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}\right] < \frac{1}{\sqrt[3]{n}} < \frac{3}{2}\left[\sqrt[3]{n^2}-\sqrt[3]{(n+1)^2}\right]$$

Answer: 14 996.

230. **(a)** 0.105. **(b)** Use the inequality

$$\frac{1}{10!}+\frac{1}{11!}+\frac{1}{12!}+\ldots+\frac{1}{1000!}<\frac{1}{9}\left\{\frac{9}{10!}+\frac{10}{11!}+\frac{11}{12!}+\ldots+\frac{999}{1000!}\right\}$$

Answer: 0.00000029.

231. Use the result of Problem 227.

232. Start with determining the number of those terms lying between $1/10^k$ and $1/10^{k+1}$ that are not deleted.

233. **(a)** This problem is solved by analogy with Problem 231. **(b)** Use the relation

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{(n-1)n}=1-\frac{1}{n}$$

234. Prove that

$$\log\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^k}\right)<\frac{2\log 3}{p}$$

for any positive integers k and $p \geqslant 2$. Proceeding from this inequality derive the inequality

$$\log\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}\right)\leqslant$$
$$\leqslant 2\log 3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p_l}\right)$$

where p_l is the greatest prime number among the numbers from 1 to n.

235. Take into account the identity $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

236. Make use of the relations $a^{10}+a^5+1=((a^5)^3-1)/(a^5-1)$ and $a^{15}-1=(a^3)^5-1$.

237. Prove that the difference $(x^{9999}+x^{8888}+\ldots+x^{1111}+1)-(x^9+x^8+\ldots+x+1)$ is divisible by $x^9+x^8+\ldots+x+1$.

238. **(a)** The expression $a^3+b^3+c^3-3abc$ is divisible by $a+b+c$. **(b)** *Answer*: $x_1=-a-b$ where

$$a,\ b=\sqrt[3]{\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

239. Eliminate radicals and solve the resultant equation with respect to a.

240. Use the fact that if $x^2+2ax+1/16=y$ then $x=-a+\sqrt{a^2+y-1/16}$; consider the graphs of the functions $y=x^2+2ax+1/16$ and $y_1=-a+\sqrt{a^2+x-1/16}$.

241. Prove that there must be $3x=x^2$.

242. **(a)** Prove that there must be $1/(1+x)=x$.

243. The roots of the equation are all the numbers lying between 5 and 10.

244. The roots of the equation are the number -2 and all the numbers not exceeding 2.

245. $x_1=1,\ x_2=2,\ \ldots,\ x_n=n$.

246. $x=\sqrt[3]{4}$.

247. For $a = \pm 1$ the system possesses three solutions; for $a = \pm\sqrt{2}$ the system possesses two solutions.

248. **(a)** For $a = -1$ the system has no solutions; for $a = 1$ the system has infinitely many solutions. **(b)** For $a = \pm 1$ the system has infinitely many solutions. **(c)** For $a = 1$ the system has infinitely many solutions; for $a = -2$ it has no solutions at all.

249. For the system to possess solutions it is necessary that three of the four numbers α_1, α_2, α_3 and α_4 should be equal to one another. If

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha, \quad \alpha_4 = \beta \quad \text{then} \quad x_1 = x_2 = x_3 = \frac{\alpha^2}{2}, \quad x_4 = \alpha\left(\beta - \frac{\alpha}{2}\right)$$

250. There is only one real solution: $x = 1$, $y = 1$, $z = 0$.

251. $x = 1$, $y = 0$ and $x = 0$, $y = 1$.

252. $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ and x is arbitrary; $x_1 = x_2 = x_3 = x_4 = x_5$ are arbitrary and $x = 2$; $x = (-1 \pm \sqrt{5})/2$, x_1 and x_2 are arbitrary, $x_3 = xx_2 - x_1$, $x_4 = -x(x_1 + x_2)$ and $x_5 = xx_1 - x_2$.

253. Either all the numbers are equal to 1 or three of them are equal to -1 and the fourth one is equal to 3.

254. For $a > b > c > d$ we have $x = t = 1/(a - d)$, $y = z = 0$.

255. Note that depending on the sign of the discriminant $\Delta = (b - 1)^2 - 4ac$ the quadratic trinomial $a\xi^2 + (b - 1)\xi + c$ either retains sign for all ξ or turns into zero for a single value of ξ or possesses two different roots $\xi = \xi_1$ and $\xi = \xi_2$.

256. There are no solutions at all when n is even and $a_1a_3 \ldots a_{n-1} \neq a_2a_4 \ldots a_n$; there are infinitely many solutions when n is even and $a_1a_3 \ldots a_{n-1} = a_2a_4 \ldots a_n$; there are two solutions when n is odd.

257. **(a)** The number of real roots of the equation coincides with the number of the points of intersection of the sine curve $y = \sin x$ and the straight line $y = x/100$. **(b)** The number of the roots is equal to the number of the points of intersection of the graphs of the functions $y = \sin x$ and $y = \log x$.

258. All the numbers $a_1, a_2, \ldots, a_{100}$ are equal.

259. Investigate the coefficients of the equation

$$P(x) = (x - a)(x - b)(x + d) = 0$$

260. Consider the reciprocal of the fraction indicated in the condition of the problem; eliminate radicals in the denominator of the resultant fraction.

261. Denoting the numbers in question as a, b and $1/ab$ we can express the assertion stated in the condition of the problem in the form $a + b + 1/ab > 1/a + 1/b + ab$.

262. Prove (using the induction method) that for any natural numbers n and k the fact that a sum of n positive numbers is equal to 1 implies that the sum of all the possible products of k (where $1 < k \leqslant n$) numbers chosen from the given numbers is less than 1.

263. 1/2.

264. All the numbers a_i (where $i = 1, 2, \ldots, 1973$) are equal (consider separately the cases $a_1 > 1$ and $a_1 < 1$).

265. Apply the method of mathematical induction.

266. This is impossible (for the second polynomial the answer to the question is positive).

267. Use the method of mathematical induction.

268. If $99\,999 + 111\,111\sqrt{3} = (A + B\sqrt{3})^2$ then $99\,999 - 111\,111\sqrt{3} = (A - B\sqrt{3})^2$.

269. Prove that the assumption that $\sqrt[3]{2} = p + q\sqrt{r}$ leads to the wrong conclusion that $\sqrt[3]{2}$ is a rational number.

270. Let $A = x^m$; then $x + 1/x = n$.

271. There are no such numbers.

272. This is impossible.

273. For $x = (6k+5)/(3-k^2)$ where k is a rational number.

274. Estimate the difference $y_1 - x_1$ between those roots of the equations

$$x^2 + px + q = 0 \quad \text{and} \quad y^2 + py + q_1 = 0 \quad \text{where} \quad |q_1 - q| \approx 0.01$$

which are close to each other.

275. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$ denote the fractional parts of the given numbers $a_1, a_2 \dots, a_n$; take the *minor* approximations of the numbers $a_1, a_2, \dots, a_n$ and the *major* approximations of the numbers $a_{k+1}, a_{k+2}, \dots, a_n$ after which choose an appropriate value of the index k.

276. 0; 0.5; 0.501; 0.502; ...; 0.999; 1.

277. Consider the fractional parts of the numbers $0, \alpha, 2\alpha, 3\alpha, \dots, 1000\alpha$ and use Dirichlet's principle (see page 9).

278. (a) Prove that if $\alpha < 1 - (0.1)^{100}$ then also $\sqrt{\alpha} < 1 - (0.1)^{100}$. (b) Take into account that the number x under consideration is equal to $(\sqrt{1-(1/10)^{100}})/3$. *Answer*: with an accuracy of 300 decimal places we have

$$x = 0.\underbrace{3333\dots333}_{100\text{ threes}}\underbrace{166666\dots6666}_{100\text{ sixes}}25\underbrace{0000\dots000}_{97\text{ noughts}}$$

279. The second of the two given expressions is greater than the first one.

280. $x = (a_1 + a_2 + \dots + a_n)/n$.

281. (a) a_1, a_2, a_4, a_3. (b) Prove that if a_{i_α} and a_{i_β} are some two of the given numbers (where $\alpha < \beta$) and if $a_{i_{\alpha+1}}$ and $a_{i_{\beta+1}}$ are the numbers preceding a_{i_α} and following a_{i_β} respectively in the sought-for arrangement then

$$\left(a_{i_\alpha} - a_{i_\beta}\right)\left(a_{i_{\alpha-1}} - a_{i_{\beta+1}}\right) > 0$$

282. (a) Consider a broken line $A_0A_1A_2\dots A_n$ such that the projections of the line segments $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$ on the axis Ox are equal to $a_1, a_2, \dots, a_n$ respectively and the projections on the axis Oy are equal to $b_1, b_2, \dots, b_n$. The equality takes place when $a_1/b_1 = a_2/b_2 = \dots = a_n/b_n$. (b) Make use of the inequality of Problem 282 (a).

283. For an even n the problem can be solved geometrically by analogy with the solution of Problem 282 (a); the case when n is odd can be reduced to the former case of an even n. The equality takes place for the even values of n when $a_1 = 1 - a_2 = a_3 = 1 - a_4 = \dots = a_{n-1} - a_n$ and for the odd values of n it holds only when $a_1 = a_2 = \dots = a_n = 1/2$.

284. Square both members of the inequality.

285. The expression $\cos\sin x$ is greater than $\sin\cos x$ for any x.

286. (a) Denoting $\log_2 \pi = a$ and $\log_5 \pi = b$ we can write $\pi^{1/a+1/b} = 10$. (b) Denoting $\log_2 \pi = a$ and $\log_\pi 2 = b$ we can write $b = 1/a$.

287. (a) Take into account that $\sin x < x$ for every angle x lying in the first quadrant. (b) Take into account that $\tan x > x$ for every angle x lying in the first quadrant.

288. Take into account that the tangents of angles can be defined geometrically in terms of their line values (with the aid of unit circle used in trigonometry) and can also be interpreted as twice the areas of some triangles.

289. $\arcsin\cos\arcsin x + \arccos\sin\arccos x = \pi/2$.

290. Replace the angle x by $x + \pi$ in the sum $\cos 32x + a_{31}\cos 31x + \dots + a_1\cos x$ and add the resultant expression to the original sum.

291. Using the formula $2\sin\alpha/2 = \pm\sqrt{2 - 2\cos\alpha}$ compute consecutively for $n = 1, 2, \dots$ the expressions

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \dots + \frac{a_1a_2\dots a_n}{2^{n-1}}\right)\cdot 45^\circ$$

292. 1.
293. Take into account that the two given polynomials and the polynomials $(1+x^2+x^3)^{1000}$ and $(1-x^2-x^3)^{1000}$ have the same coefficients in x^{20} respectively.
294. Make use of the formula $(a+b)(a-b)=a^2-b^2$.
295. **(a)** $C(1001, 50) = 1001!/50!\cdot 951!$. **(b)** $1000\,C(1001, 51) - C(1001, 52) = 51\,050\cdot 1001!/52!\cdot 950!$
296. Let us denote the given expression as Π_k; then $\Pi_k = (\Pi_{k-1}-2)^2$. *Answer*: $(4^{2k-1}-4^{k-1}):3$.
297. **(a)** 6. **(b)** $6x$.
298. $-x+3$.
299. Use the fact that the polynomial $x^4+x^3+2x^2+x+1$ is a divisor of the binomial $x^{12}-1$. *Answer*: -1.
300. $P(x)=cx(x-1)(x-2)\ldots(x-26)$ where c is constant.
301. **(a)** Consider the numbers $P(10^N)$ for sufficiently large N. **(b)** Use the result of Problem 301 (a).
302. In the equality $x^{200}y^{200}+1=f(x)g(y)$ first put $y=0$ and then $x=0$.
303. Take into account that the quadratic trinomial $p(x)-x$ retains sign for all x.
304. Make use of the inequalities $|p(1)|\leqslant 1$, $|p(0)|\leqslant 1$ and $|p(-1)|\leqslant 1$.
305. Consider the two numbers $p(x_1)$ and $p(x_2)$ where $p(x)$ is the polynomial on the left-hand side of equation (3).
306. $Q^2+q^2-pP(Q+q)+qP^2+Qp^2-2Qq$.
307. $a=1$ and $a=-2$.
308. **(a)** $a=8$ and $a=12$. **(b)** $b=1$, $c=2$, $a=3$; $b=-1$, $c=-2$, $a=-3$; $b=2$, $c=-1$, $a=1$ and $b=1$, $c=-2$, $a=-1$.
309. **(a)** The representation is impossible. **(b)** Only when $n=2$, $a_2=a_1+2$ and $n=4$; $a_2=a_1-1$, $a_3=a_1+1$, $a_4=a_1+2$,
310. Use the fact that if

$$(x-a_1)^2(x-a_2)^2\ldots(x-a_n)^2+1=p(x)\,q(x)$$

then the polynomials $p(x)$ and $q(x)$ as well as the product $(x-a_1)^2(x-a_2)^2\ldots(x-a_n)^2+1$ cannot turn into zero for any x and therefore cannot change sign. In all the other respects the solution is quite analogous to that of Problem 309 (a).
311. Take into account that the number $14-7=7$ cannot be expressed as a product of several integral factors among which four factors are different.
312. Use the fact that if the given polynomial can be expressed as a product of polynomials with integral coefficients then the values of x for which the polynomial is equal to ± 1 also turn the latter polynomials into ± 1 and that a polynomial of the third degree cannot assume one and the same value more than three times.
313. Take into account that if p and q are two integers then $P(p)-P(q)$ is divisible by $p-q$.
314. Prove that if $P(k/l)=0$ then $k-pl=\pm 1$ and $k-ql=\pm 1$.
315. **(a)** Equate the coefficients in like powers of x on both sides of the equality

$$(a_0+a_1x+a_2x^2+\ldots+a_nx^n)(b_0+b_1x+b_2x^2+\ldots+b_nx^n)=$$
$$=c_0+c_1x+c_2x^2+\ldots+c_{n+m}x^{n+m}$$

and, using the resultant formula, show that if the given polynomial could be expressed as a product of two polynomials with integral coefficients then all the coefficients of one of the polynomial factors must be even (which is impossible because the leading coefficient of the original polynomial is equal to 1).

(b) Put $x = y + 1$ in the given polynomial and then, by analogy with the solution of Problem 315 (a), show that if the resultant polynomial could be expressed as a product of two polynomial factors with integral coefficients then all the coefficients of one of the factors must be divisible by the prime number 251.

316. Use the same formula as the one in the solution of Problem 315 (a) (see the hint to the latter problem).

317. Prove that the expression $P(p/q)$ where p/q is an irreducible fraction cannot be equal to an integral number.

318. Let $P(N) = M$; prove that $P(N + kM) - P(N)$ is divisible by M for any k.

319. Write the polynomial $P(x)$ which assumes integral values for the integral values of x in the form of a sum $P(x) = b_0P_0(x) + b_1P_1(x) + \ldots + b_nP_n(x)$ with indeterminate coefficients $b_0, b_1, \ldots, b_n$ where $P_k(x) = C(x, k)$ (for integral $x \geqslant k$) and then determine these coefficients by substituting consecutively into the last equality the values $x = 0, 1, 2, 3, \ldots, n$.

320. **(a)** See the hint to the foregoing problem. **(b)** Perform the change of variable $y = x + k$ in the given polynomial. **(c)** Consider the polynomial $Q(x) = P(x^2)$.

321. Use De Moivre's formula.

322. Make use of the result of Problem 321 (b).

323. If we put $x + 1/x = 2\cos\alpha$ then $x = \cos\alpha \pm i \sin\alpha$.

324. Use De Moivre's formula.

325. Make use of the result of the foregoing problem. *Answer*:

$$\cos^2\alpha + \cos^2 2\alpha + \ldots + \cos^2 n\alpha = \frac{n-1}{2} + \frac{\sin(n+1)\alpha\cos n\alpha}{2\sin\alpha}$$

326. Use De Moivre's formula and Newton's binomial formula.

327. Apply the formula

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

and use the results of Problem 324.

328. Consider the roots of the equation $x^{2n+1} - 1 = 0$.

329. Make use of the formulas of Problem 321 (b).

330. Make use of the result of Problem 329 (b). *Answer*: **(a)** $n(2n - 1)/3$. **(b)** $2n(n + 1)/3$.

331. Make use of the result of Problem 329 (a). *Answer*: **(a)** $\sqrt{2n+1}/2^n$ and $\sqrt{n}/2^{n-1}$. **(b)** $1/2^n$ and $\sqrt{n}/2^{n-1}$.

332. Take into account that for an angle α lying in the first quadrant we always have $\sin\alpha < \alpha < \tan\alpha$.

333. **(a)** and **(b)** Use the formula of Problem 324. **(c)** Use the result of Problem (b).

334. **(a)** Use the proposition established in Problem (a). **(b)** Make use of the formula of Problem 324.

335. **(a)** Use the proposition established in Problem 333 (a). **(b)** See the hint to Problem 334 (b). **(c)** Use the result of Problem 331 (a).

336. Using De Moivre's formula represent $\sin^{50}\alpha$ in the form of a sum of products of cosines of angles multiple of α by some coefficients. *Answer*: $5000\, C(50, 25)R^{50} = (5000 \cdot 50!\, R^{50}) : (25!)^2$

337. The greatest value is $|z| = (a + \sqrt{a^2 + 4})/2$; the smallest value is $|z| = (\sqrt{a^2 + 4} - a)/2$.

338. Prove that if the greatest of the differences between the arguments of the given numbers is less than 120° then they can be multiplied by a number with unit absolute value such that the real part of the sum of the resultant products is positive. It is impossible to replace the value 120° indicated in the condition of the problem by a greater value

339. Use the fact that if the point A representing the number z in the complex plane lies outside the polygon $\mathbf{M} = C_1C_2 \ldots C_n$ whose vertices represent the numbers $c_1, c_2 \ldots, c_n$ then A "lies on one side" of $\mathbf{M}$ in the sense that all the vectors AM where $M \in \mathbf{M}$ go in one direction from a straight line l passing through A.

340. Take into account that if a is not divisible by p then the division of the numbers $a, 2a, 3a, \ldots, (p-1)a$ by p leaves different remainders.

341. Take into account that if $k_1, k_2 \ldots, k_r$ are all positive integers smaller than N and relatively prime to N then the division of the numbers $k_1a, k_2a, \ldots \ldots, k_ra$ by N leaves different remainders.

342. Make use of the principle of mathematical induction.

343. Apply Euler's theorem (see Problem 341).

344. Using the induction method prove that for any whole number N there always exists a power of the number 2 the last N digits of whose decimal representation are all unities and twos; to elaborate the proof make use of Euler's theorem (see Problem 341) and of the proposition of Problem 342.

345. It is clear that a pair of numbers n and n^2 is "good"; compare the factorizations of the numbers $n-1$ and n^2-1.

346. Let a and d be relatively prime; using Euler's theorem (see Problem 341) prove that the progression contains infinitely many powers of the number a with natural exponents.

347. See the hint to Problem 340.

348. **(a)** Prove that for every odd prime number p there exist two positive integers x and y (where $x, y < p/2$) such that the division of x^2 and y^2-1 by p leaves equal remainders. **(b)** Make use of Wilson's theorem (see Problem 347).

349. Let $p_1, p_2, \ldots, p_n$ be n prime numbers. Find a number which is divisible by neither of these numbers and is greater than each of them.

350. **(a)** Let $p_1, p_2, \ldots, p_n$ be n prime numbers of the form $4k-1$ (or of the form $6k-1$). Find a number of the form $4N-1$ (or, accordingly, of the form $6N-1$) which is divisible by neither of the numbers $p_1, p_2, \ldots, p_n$ and is greater than each of them. **(b)** The idea of the solution of this problem is close to the one on which the solution of Problem 350 (a) is based.

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