Linear and Quadratic Equations

Dr. Sankhadip Chakraborty

0.1 Lesson Overview

Polynomial equations of small degrees are the first objects that one encounters in Algebra. In this lesson, we shall discuss the question of their solvability and also see explicit expressions for the solutions.

0.2 First degree equations

A first degree equation is an equation of the form

$$ax + b = 0,$$

where $a \neq 0$. It is trivial to see that such an equation has a unique solution, namely $x = -\frac{b}{a}$. Geometrically speaking, it is the intersection of the line $y = ax + b$ with the X axis. When $a = 0$, the line is parallel to the X axis and hence there is no intersection.

0.3 Second degree equations

A second degree equation is an equation is an equation of the form

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. It has 0, 1 or 2 solutions.

Note that the equation can be rewritten as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. The idea is to manipulate the terms to complete a square. This can be achieved be adding and subtracting $\frac{b^2}{4a^2}$:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= x^2 + 2 \times \frac{b}{2a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} \\ &= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}. \end{aligned}$$

Now there are three cases to consider :

- $b^2 - 4ac < 0$

  In this case, we have $\left(x + \frac{b}{2a}\right)^2 < 0$ which cannot be satisfied by any real number x. Thus there are no solutions in this case.

- $b^2 - 4ac = 0$

  This corresponds to $\left(x + \frac{b}{2a}\right)^2 = 0$. This has the unique solution $x = -\frac{b}{2a}$.

- $b^2 - 4ac > 0$

  It is easy to see that there are two distinct solutions, given by $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Mathematical tidbits

In a similar manner, it is possible define equations of higher degrees. A polynomial equation of degree n is an equation of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$. Such equations always have explicit solutions in terms of the coefficients for degrees 1, 2, 3 and 4. However, no such formula exists for degree greater than or equal to 5. This is a very profound result that had baffled mathematicians for centuries before finally being proven.