Cheenta Academy
FOR OLYMPIADS AND RESEARCH

CMI BSc Entrance 2016

5. Find a polynomial $p(x)$ that simultaneously has both the following properties.
(i) When $p(x)$ is divided by x^{100} the remainder is the constant polynomial 1.
(ii) When $p(x)$ is divided by $(x-2)^3$ the remainder is the constant polynomial 2.

Division of Polynomials:

Given polynomials $f(x)$ & $g(x)$, $\exists$ polynomials $q(x)$ & $r(x)$ s.t.

$$f(x) = g(x)q(x) + r(x)$$

where $0 \le \deg r(x) < \deg g(x)$

$$x^2+1 = x \cdot x + 1$$

$$\boxed{\begin{aligned} & p(x) \equiv 1 \mod x^{100} \\ \& \ & p(x) \equiv 2 \mod (x-2)^3 \end{aligned}}$$

Generally, if $f(x)$ when divided by $(x-c)^k$, leaves a const. remainder, then $f'(x)$ is divisible by $(x-c)^{k-1}$

$$f(x) = (x-c)^k \cdot q(x) + \underline{r}$$

$$\Rightarrow f'(x) = \left((x-c)^k q'(x) + k(x-c)^{k-1} q(x) \right)$$

$$= \underline{(x-c)^{k-1}} \left((x-c) q'(x) + k q(x) \right)$$

If $f'(x)$ is divisible by $(x-c)^{k-1}$

$\Leftarrow)$ $f(x)$ leaves a const. remainder when divided by $(x-c)^k$

$$f'(x) = (x-c)^{k-1} q(x)$$

Dif.

$$\Rightarrow f(x) + c = \int u^{k-1} q(u+c)\, du$$

By FTC

$$u^k(\text{—})$$

$p'(x)$ divisible by $x^{99} \times (x-2)^2$

$$x \cdot x \cdots x \qquad (x-2)(x-2)$$

$\Rightarrow p'(x)$ is a multiple $x^{99}(x-2)^2$

$$p'(x) = A x^{99}(x^2 - 4x + 4)$$

$$\Rightarrow p(x) = A\left(\frac{x^{102}}{102} - \frac{4 \cdot x^{101}}{101} + \frac{4x^{100}}{100}\right) + B$$

Remainder Thr.

$g(x)$ is divided by $(x-a)^k$, the remainder is $g(a)$

$$p(0) = 1$$

$$p(2) = 2$$

$$p(0) = B = 1 \Rightarrow B = 1$$

$$p(2) = 2 \Rightarrow A\left(\frac{2^{102}}{102} - \frac{4 \cdot 2^{101}}{101} + \frac{4 \cdot 2^{100}}{100}\right) + 1 = 2$$

$$\Rightarrow A = \frac{1}{(\qquad\qquad)}$$

$x \equiv r \bmod a$ when a & b are coprime

$x \equiv s \bmod b$

Euclidean Algorithm:

x_0 & $y_0 \in \mathbb{Z}$ s.t.

$$a\underline{x_0} + b\underline{y_0} = 1 \quad (\gcd(a,b) = 1)$$

$$\Rightarrow rby_0 \equiv r \bmod a$$

$$\& \; sax_0 \equiv s \bmod b.$$

$$\Rightarrow rby_0 + sax_0 = x$$

Run EA backwards to find $p_1(x)$ & $p_2(x)$ s.t.

$$p_1(x) \cdot x^{100} + p_2(x) \cdot (x-2)^3 = 1$$

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